

Quantum Lovász Local Lemma: Shearer’s Bound is Tight

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Abstract

Lovász Local Lemma (LLL) is a very powerful tool in combinatorics and probability theory to show the possibility of avoiding all “bad” events under some “weakly dependent” condition. Over the last decades, the algorithmic aspect of LLL has also attracted lots of attention in theoretical computer science [23, 28, 35]. A tight criterion under which the *abstract* version LLL (ALLL) holds was given by Shearer [44]. It turns out that Shearer’s bound is generally not tight for *variable* version LLL (VLLL) [24]. Recently, Ambainis et al. [3] introduced a quantum version LLL (QLLL), which was then shown to be powerful for the quantum satisfiability problem.

In this paper, we prove that Shearer’s bound is tight for QLLL, i.e., the relative dimension of the smallest satisfying subspace is completely characterized by the independent set polynomial, affirming a conjecture proposed by Sattath et al. [34, 39]. Our result also shows the tightness of Gilyén and Sattath’s algorithm [18], and implies that the lattice gas partition function fully characterizes quantum satisfiability for almost all Hamiltonians with large enough qudits [39].

Commuting LLL (CLLL), LLL for commuting local Hamiltonians which are widely studied in the literature, is also investigated here. We prove that the tight regions of CLLL and QLLL are different in general. This result might imply that it is possible to design an algorithm for CLLL which is still efficient beyond Shearer’s bound.

In applications of LLLs, the symmetric cases are most common, i.e., the events are with the same probability [15, 16] and the Hamiltonians are with the same relative dimension [3, 39]. We give the first lower bound on the gap between the symmetric VLLL and Shearer’s bound. Our result can be viewed as a quantitative study on the separation between quantum and classical constraint satisfaction problems. Additionally, we obtain similar results for the symmetric CLLL. As an application, we give lower bounds on the critical thresholds of VLLL and CLLL for several of the most common lattices.

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1 Introduction

Classical Lovász Local Lemma *Lovász Local Lemma* (or LLL) is a very powerful tool in combinatorics and probability theory to show the possibility of avoiding all “bad” events under some “weakly dependent” condition, and has numerous applications. Formally, given a set \mathcal{A} of bad events in a probability space, LLL provides the condition under which $\mathbb{P}(\cap_{A \in \mathcal{A}} \bar{A}) > 0$. The dependency among events is usually characterized by the dependency graph. A dependency graph is an undirected graph $G_D = ([m], E_D)$ such that for any vertex i , A_i is independent of $\{A_j : j \notin \Gamma_i \cup \{i\}\}$, where Γ_i stands for the set of neighbors of i in G_D . In this setting, finding the conditions under which $\mathbb{P}(\cap_{A \in \mathcal{A}} \bar{A}) > 0$ is reduced to the following problem: given a graph G_D , determine its abstract interior $\mathcal{I}(G_D)$ which is the set of vectors \mathbf{p} such that $\mathbb{P}(\cap_{A \in \mathcal{A}} \bar{A}) > 0$ for any event set \mathcal{A} with dependency graph G_D and probability vector \mathbf{p} . *Local* solutions to this problem, including the first LLL proved in 1975 by Erdős and Lovász [11], are referred as abstract-LLL (or ALLL).

The most frequently used abstract-LLL is as follows:

Theorem 1.1 ([45]). *Given a dependency graph $G_D = ([m], E_D)$ and a probability vector $\mathbf{p} \in (0, 1)^n$, if there exist real numbers $x_1, \dots, x_n \in (0, 1)$ such that $p_i \leq x_i \prod_{j \in \Gamma_i} (1 - x_j)$ for any $i \in [m]$, then $\mathbf{p} \in \mathcal{I}(G_D)$.*

Shearer [44] provided the exact characterization of $\mathcal{I}(G_D)$ with the independence polynomial defined as follows.

Definition 1.1 (Multivariate independence polynomial). Let $G_D = (V, E)$, $\mathbf{x} = (x_v : v \in V)$ and let $\text{Ind}(G_D)$ be the set of all independent sets of G_D . Then we call $I(G_D, \mathbf{x}) = \sum_{S \in \text{Ind}(G_D)} (-1)^{|S|} \prod_{v \in S} x_v$ the *multivariate independence polynomial*.

Definition 1.2 (Shearer’s bound). A probability vector $\mathbf{p} = (p_v : v \in V) \in \mathbb{R}^{|V|}$ is called *beyond Shearer’s bound* for a dependency graph G_D if there is a vertex set $V' \subseteq V$ such that for the corresponding induced subgraph $G_D(V') := (V', E')$: $I(G_D', (p_v : v \in V')) \leq 0$. Otherwise we say \mathbf{p} is *in Shearer’s bound* for G_D .

The tight criterion under which *abstract* version LLL holds provided by Shearer is as follows.

Theorem 1.2 ([44]). *For a dependency graph $G_D = (V, E)$ and probabilities $\mathbf{p} \in \mathbb{R}^{|V|}$ the following conditions are equivalent:*

1. \mathbf{p} is in Shearer’s bound for G_D .
2. for any probability space Ω and events $\{A_v \subseteq \Omega : v \in V\}$ having G_D as dependency graph and satisfying $\mathbb{P}(A_v) \leq p_v$, we have $\mathbb{P}(\overline{\cup_{v \in V} A_v}) \geq I(G_D, \mathbf{p}) > 0$.

In other words, $\mathbf{p} \in \mathcal{I}(G_D)$ if and only if \mathbf{p} is in Shearer’s bound for G_D .

Another important version of LLL, *variable version Lovász Local Lemma* (or VLLL), which exploits richer dependency structures of the events, has also been studied [24, 28]. In this setting, each event A_i can be fully determined by some subset \mathcal{X}_i of a set of mutually independent random variables $\mathcal{X} = \{X_1, \dots, X_n\}$. Thus, the dependency can be naturally characterized by the event-variable graph defined as follows. An event-variable graph is a bipartite graph $G_B = ([m], [n], E)$ such that for any $X_j \in \mathcal{X}_i$, there is an edge $(i, j) \in [m] \times [n]$. Similar to the abstract-LLL, the VLLL is for solving the following problem: given a bipartite graph G_B , determine its variable interior $\mathcal{VI}(G_B)$ which is the set of vectors

\mathbf{p} such that $\mathbb{P}(\cap_{A \in \mathcal{A}} \bar{A}) > 0$ for any variable-generated event system \mathcal{A} with event-variable graph G_B and probability vector \mathbf{p} .

The VLLL is important because many problems in which LLL has applications naturally conform with the variable setting, including hypergraph coloring [32], satisfiability [15, 16], counting solutions to CNF formulas [33], acyclic edge coloring [19], etc. Moreover, most of recent progresses on the algorithmic aspects of LLL are based on the variable model [28, 35, 37].

A key problem around the VLLL is whether Shearer's bound is tight for variable-LLL [28]. Formally, given a bipartite graph $G_B = (U, V, E)$, its *base graph* is defined as the graph $G_D(G_B) = (U, E')$ such that for any two nodes $u_i, u_j \in U$, there is an edge $(u_i, u_j) \in E'$ if and only if u_i and u_j share some common neighbor in G_B . That is to say, $G_D(G_B)$ is a dependency graph of the variable-generated event system with event-variable graph G_B . Thus, we have $\mathcal{I}(G_D(G_B)) \subseteq \mathcal{VI}(G_B)$ immediately. If $\mathcal{I}(G_D(G_B)) \neq \mathcal{VI}(G_B)$, we say that Shearer's bound is not tight for G_B , or G_B has a gap. The first example of gap existence is a bipartite graph whose base graph is a cycle of length 4 [28]. Recently, He et al. [24] have shown that Shearer's bound is generally not tight for variable-LLL. More precisely, Shearer's bound is tight if the base graph G_D is a tree, while not tight if G_D has an induced cycle of length at least 4. The remaining case when G_D has only 3-cliques is partially solved.

Quantum Satisfiability and Quantum Lovasz Local Lemma Most systems of physical interest can be described by local Hamiltonians $H = \sum_i H_i$ where each k -local term H_i acts nontrivially on, at most, k qudits. We say H is frustration free if the ground state $|\phi\rangle$ of H is also the ground state of every H_i . Let Π_i be the projection operator on the excited states of H_i and $\Pi = \sum \Pi_i$, and it is easy to see that the frustration freeness of H and Π are the same. Henceforth, we only care about the Hamiltonians that are projectors. Determining whether a given Π is frustration free (or satisfiable, in computer science language), known as the quantum satisfiability problem, is a central pillar in quantum complexity theory, and has many applications in quantum many-body physics.

Unfortunately, the quantum satisfiability problem has been shown to be QMA_1 -complete [5], which is widely believed to be intractable in general even for quantum computing. This makes it highly desirable to search for efficient heuristics and algorithms in order to, at least, partially answer this question.

In the seminal paper, by generalizing the notations of probability and independence as described in the following table, Ambainis et al. [3] introduced a quantum version LLL (or QLLL) with respect to the dependency graph, i.e., a sufficient condition under which the Hamiltonian is guaranteed to be frustration free given relative dimensions. Here, the relative dimension of a Hamiltonian is defined as that of the subspace it projects. With QLLL, they [3] greatly improved the known critical density for random k -QSAT from $\Omega(1)$ [29] to $\Omega(2^k/k^2)$, almost meeting the best known upper bound of $O(2^k)$ [29].

Probability space Ω	\rightarrow Vector space V
Event A	\rightarrow Subspace $A \subseteq V$
Complement $\bar{A} = \Omega \setminus A$	\rightarrow Orthogonal complementary subspace A^\perp
Probability $\mathbb{P}(A)$	\rightarrow Relative dimension $R(A) := \frac{\dim(A)}{\dim(V)}$
Disjunction $A \vee B$	$\rightarrow A + B = \{a + b a \in A, b \in B\}$
Conjunction $A \wedge B$	$\rightarrow A \cap B$
Independence $\mathbb{P}(A \wedge B) = P(A) \cdot P(B)$	$\rightarrow R(A \cap B) = R(A) \cdot R(B)$
Conditioning $\mathbb{P}(A B) = \frac{\mathbb{P}(A \wedge B)}{\mathbb{P}(B)}$	$\rightarrow R(A B) := \frac{R(A \cap B)}{R(B)}$

Recently, Sattath et al. [39] generalized Shearer's theorem to QLLL with respect to the interaction

bipartite graph, and showed that Shearer’s bound is still a sufficient condition here. Here, the interaction bipartite graph is the quantum analog of the classical event-variable graph, where the left vertices represent Hamiltonians, the right vertices represent qudits, and an edge between a left and right vertex means the corresponding Hamiltonian acts on the corresponding qudit. Remarkably, the probability threshold of Shearer’s bound turns out to be the first negative fugacity of the hardcore lattice gas partition function, which has been extensively studied in classical statistical mechanics. Utilizing the tools in classical statistical mechanics, they concretely apply QLLL to evaluate the critical threshold for various regular lattices. In contrast to VLLL [24] which generally goes beyond Shearer’s bound, Sattath et al. [39] conjectured that Shearer’s bound is tight for QLLL, which, if true, would have important physical significance and several striking consequences [39].

In the past few years, as a special case of the quantum satisfiability problem, the commuting local Hamiltonian problem (CLH), where $[\Pi_i, \Pi_j] = 0$ for all i and j , has attracted considerable attention [1, 2, 7, 20, 40]. Commuting Hamiltonians are somewhat “halfway” between classical and quantum, and are capable of exhibiting intriguing multi-particle entanglement phenomena, such as the well-known toric code [27]. CLH interests people not only because the commutation restriction is natural and often made in physics, but also because it may help us to understand the centrality of non-commutation in quantum mechanics. CLH can be viewed as a generalization of the classical SAT, thus CLH is at least NP-hard, and as a sufficient condition, the commuting version LLL (or CLLL) is desirable and would have various applications.

The QLLLs provide sufficient conditions for frustration freeness. A natural question is whether there is an efficient way to prepare a frustration-free state under the conditions of QLLL. A series of results showed that the answer is affirmative if all local Hamiltonians commute [10, 38, 41]. Recently, Gilyén and Sattath improved the previous constructive results by designing an algorithm that works efficiently under Shearer’s bound for non-commuting terms as well under the condition that the Hamiltonian has a uniform inverse polynomial gap. Here, a uniform gap is the minimum energy gap among the system and all its subsystems [18].

Therefore, the following three closely related problems beg answers:

1. Tight region for QLLL: complete characterization of the interior of QLLL, $QI(G_B)$, for a given interaction bipartite graph G_B . Here the interior $QI(G_B)$ is the set of vectors \mathbf{r} such that any local Hamiltonians with relative dimensions \mathbf{r} and interaction bipartite graph G_B are frustration free. As Shearer’s bound has been shown to be a sufficient condition for QLLL [39], a fundamental open question here is whether Shearer’s bound is tight. If it is tight, there are several striking consequences. First, the tightness implies that Gilyén and Sattath’s algorithm [18] converges up to the tight region assuming a uniform inverse-polynomial spectral gap of the Hamiltonian. Second, the geometrization theorem [30] says that given the interaction bipartite graph, dimensions of qudits, and dimensions of local Hamiltonians, either all such Hamiltonian are frustration free, or almost all such Hamiltonians are not. If Shearer’s bound is indeed tight for QLLL, by geometrization theorem we know that the quantum satisfiability for almost all Hamiltonians with large enough qudits can be completely characterized by the lattice gas partition function. The lattice gas critical exponents can be directly applied to count of the ground state entropy of almost all quantum Hamiltonians in the frustration free regime. Thus, the tightness means a lot for transferring insights from classical statistical mechanics into the quantum complexity domain [39].
2. Tight region for CLLL: complete characterization of the interior of CLLL, $CI(G_B)$, for a given interaction bipartite graph G_B . Here the interior $CI(G_B)$ is the set of vectors \mathbf{r} such that any *com-*

muting Hamiltonians with relative dimensions \mathbf{r} and interaction bipartite graph G_B are frustration free. It is immediately obvious that the interior of QLLL is a subset of the interior of CLLL for any G_B . An interesting question that remains is whether the containment is proper. There are a series of results on the algorithms for preparing a frustration-free state for commuting Hamiltonians under the conditions of QLLL [10, 38, 41]. Thus if the containment turns out to be proper, it might be possible to design a more specialized algorithm for commuting Hamiltonians that is still efficient beyond the conditions of QLLL, e.g., Shearer’s bound. The tight region for CLLL requires characterization not only due to the various applications in CLH, but also because it may help us to understand the role of non-commutation in the quantum world.

3. Critical thresholds for LLLs: determining the critical probability threshold of VLLL and the critical relative dimension thresholds of CLLL and QLLL. Here the critical thresholds of LLLs are the minimum probability p such that $\mathbb{P}(\cap_{A \in \mathcal{A}} \overline{A}) = 0$ holds for some \mathcal{A} with probability vector (p, p, \dots, p) and the minimum relative dimension r such that some $H = \sum_i H_i$ with relative dimension vector (r, r, \dots, r) is not frustration free. Rather than other boundary probability vectors or relative dimension vectors, the symmetric boundary vector where all the elements are equal is much more often considered by physicists [3, 39, 42, 46] and computer scientists [15, 16, 21, 22, 32, 33, 47]. Sattath et al. [39] conjectured that the tight regions of VLLL and QLLL are different. If this conjecture turns out to be true, the next question is how large the gap is. A lower bound on the gap between VLLL and QLLL, especially in the symmetric direction, constitutes a quantitative analysis of the relative power of quantum. Though we have the complete characterizations of LLLs, new ideas are still needed to quantify the critical thresholds and their gaps, because the mathematical characterizations, such as Shearer’s inequality system and the program for VLLL, are usually hard to solve [22, 24].

1.1 Results and Discussion

In this paper, we concentrate on the following three problems: the tight region for QLLL, the tight region for CLLL, and tight bounds for symmetric VLLL, CLLL and QLLL. We provide a complete answer for the first problem and partial answers for the other two problems. Our results show that Shearer’s bound, which is tight for abstract-LLL, is also tight for QLLL. The CLLL behaves very differently from QLLL, i.e., the interior of CLLL goes beyond Shearer’s bound generally. Moreover, we provide a lower bound on the critical thresholds of VLLL and CLLL, which are strictly larger than that of ALLL and QLLL on lattices. The main results are listed and discussed as follows.

In this work, the interaction bipartite graph of Hamiltonians and the classical event-variable graph are both denoted by the bipartite graph $G_B = ([m], [n], E)$. We call the vertices in $[m]$ the left vertices and those in $[n]$ the right vertices. Usually, we will index the left vertices with “ i ” and the right vertices with “ j ”. In G_B , there may be two vertices with the same index k : one is the left vertex and the other is the right vertex. In this paper, there will never be ambiguity in identifying which vertex is which from the context.

1.1.1 Tight Region for QLLL

Shearer’s bound is tight for QLLL In this paper, we first prove the tightness of Shearer’s bound for QLLL, which affirms the conjecture in [34, 39]. More precisely,

Theorem 1.3 (Informal). *Given an interaction bipartite graph $G_B = ([m], [n], E)$ and rational $\mathbf{r} \in (0, 1)^m$, consider the Hamiltonians Π with relative projector ranks \mathbf{r} and conforming with G_B .*

- *If $\mathbf{r} \in \mathcal{I}(G_D(G_B))$, then $R(\ker \Pi) \geq I(G_D(G_B), \mathbf{r}) > 0$ [39] for all such Hamiltonians. For qudits of proper dimensions, this lower bound can be achieved by almost all such Hamiltonians acting on these qudits. Moreover, there exists a \mathbf{d}_0 such that for all qudits with dimensions $\mathbf{d} \geq \mathbf{d}_0$, we have $R(\ker \Pi) \leq I(G_D(G_B), \mathbf{r}) + \epsilon$ for almost all such Hamiltonians, where $\epsilon > 0$ can be arbitrarily small as \mathbf{d}_0 goes to infinity.*
- *Otherwise, for qudits of proper dimensions, almost all such Hamiltonians acting on these qudits are not frustration free. Furthermore, there exists a \mathbf{d}_0 such that for all qudits with dimensions $\mathbf{d} \geq \mathbf{d}_0$, we have $R(\ker \Pi) \leq \epsilon$ for almost all such Hamiltonians, where $\epsilon > 0$ can be arbitrarily small as \mathbf{d}_0 goes to infinity.*

In contrast to the VLLL which goes beyond Shearer’s bound generally, QLLL is another example of the difference between the classical world and the quantum world. As mentioned above, Theorem 1.3 means that the position of the first negative fugacity zero of the lattice gas partition function is exactly the critical threshold of quantum satisfiability for almost all Hamiltonians with large enough qudits, and the relative dimension of the smallest satisfying subspace is exactly characterized by the independent set polynomial. Additionally, the above theorem also shows the tightness of Gilyén and Sattath’s algorithm assuming a uniform inverse polynomial spectral gap[18], which prepares a frustration free state under Shearer’s bound.

Independently, Siddhardh Morampudi and Chris Laumann showed that Shearer’s bound is tight for a large class of graphs [34]. Our result shows that Shearer’s bound is tight for any graph.

Finally, the \mathbf{d}_0 that we obtain is tremendously large (see the formal statement of Theorem 1.3 in Section 3). We are curious about how small \mathbf{d}_0 can be, and particularly whether \mathbf{d}_0 can be polynomially bounded by the vector \mathbf{r} . This open problem is important especially for the computational aspects of QLLL.

It seems [3, 6, 29, 39] that QLLL has three ranges: for sufficiently small relative ranks, there is a classical (unentangled) satisfying state, and when the relative ranks are increased the states need to become entangled in order to satisfy all Hamiltonians, just before the system becomes unsatisfiable. As only two ranges are studied in Theorem 1.3: satisfiable or unsatisfiable, it is another important open problem to investigate when the satisfying state must be entangled.

1.1.2 Tight Region for CLLL

We partially depict the tight region of CLLL. We show that Shearer’s bound is tight for CLLL on trees and explicitly provide the relative dimension bounds. On the other hand, we also show that the tight region of CLLL can go beyond Shearer’s bound if its base graph has an induced cycle of length at least 4. To obtain this result, we first prove that the tight regions of CLLL and VLLL are the same for a large family of interaction bipartite graphs (see Theorem 4.8) by Bravyi and Vyalyi’s *Structure Lemma* [7]. Then we generalize the tools for VLLL developed in [24] to CLLL, including a sufficient and necessary condition for deciding whether Shearer’s bound is tight and the reduction rules. At last, we combine these tools with the conclusions for VLLL from [24] to finish the proof.

Equal to Shearer’s bound on trees. Studies of the boundaries of LLLs on the interaction bipartite graph which is a tree, have a long history, including 1-D chains [36], regular trees [9, 25, 39, 44], and treelike bipartite graphs [24]. For LLLs on trees, our results include: 1, we prove that Shearer’s bound is tight for CLLL by the reduction rules (see Theorem 4.15); 2, we calculate the bound for CLLL explicitly even considering the dimensions of qudits (see Theorem 4.16); 3, we calculate the tight bound for LLLs explicitly ignoring the dimensions of qudits. The tight bound is as follows.

Given an interaction bipartite graph $G_B = ([m], [n], E_B)$ which is a tree, without loss of generality, we can assume that the root is the right vertex. Furthermore, it is lossless as well to assume the leaves of the tree are right vertices, because adding right vertices as leaves do not change the boundary (see Theorem 4.13).

Theorem 1.4. *For any interaction bipartite graph $G_B = ([m], [n], E_B)$ which is a tree, we have $\mathcal{VI}(G_B) = \mathcal{CI}(G_B) = \mathcal{QI}(G_B) = \mathcal{I}(G_D(G_B))$. Given $\mathbf{r} \in (0, 1)^m$, $\mathbf{r} \in \mathcal{VI}(G_B)$ if and only if there exists $\mathbf{q} \in [0, 1]^n$ where $q_j = 0$ if j is a leaf of G_B and $q_j = \sum_{i \in \mathcal{C}_j} r_i \cdot \prod_{k \in \mathcal{C}_i} \frac{1}{1 - q_k}$ for other $j \in [n]$.*

The above theorem is not an immediate corollary by Shearer’s bound. Shearer’s bound is difficult to solve in general. However, the explicit bound in the above theorem can be calculated efficiently. Sattath et al. [39] calculated the critical threshold of ALLL on the (t, k) -regular tree, which are also the critical thresholds for the other three LLLs as implied by Theorem 1.4:

Corollary 1.5. *For the interaction bipartite graph $G_B = ([m], [n], E)$ which is a (t, k) -regular tree, i.e., any left vertex is of degree k , any right vertex is of degree t and the bipartite graph is a tree, the critical thresholds of ALLL, VLLL, CLLL and QLLL are all $\frac{1}{t-1} \cdot \frac{(k-1)^{(k-1)}}{k^k}$.*

Beyond Shearer’s bound if the interaction bipartite graph contains cyclic bipartite graphs. A bipartite graph $G_B = ([l], [n], E)$ is said to be l -cyclic if the base graph $G_D(G_B)$ is a cycle of length l . When $l = 3$, it additionally requires that there is no right vertex adjacent to all three left vertices. In cases of no ambiguity, a l -cyclic bipartite graph is simply called a cyclic bipartite graph, a l -cyclic graph or a cyclic graph. We say a bipartite graph G_B contains a cyclic bipartite graph, if there are l left vertices such that the induced subgraph on these l left vertices and their neighbors (i.e., the adjacent right vertices) is a l -cyclic bipartite graph by deleting the right vertices with degree 1. By coupling our tools for CLLL with the conclusions about VLLL [24], we can prove the following theorem.

Theorem 1.6. *For any interaction bipartite graph containing a cyclic bipartite graph, the tight region of CLLL goes beyond Shearer’s bound.*

Our theorem might imply that it is possible to design a more specialized algorithm for CLLL which is still efficient beyond Shearer’s bound. Meanwhile, recall that Shearer’s bound is tight for QLLL, the above theorem shows that CLLL behaves very different from QLLL.

By Theorems 1.4 and 1.6, we can prove the following corollary, which gives an almost complete characterization of whether Shearer’s bound is tight for CLLL except when the base graph has only 3-cliques.

Corollary 1.7. *Given an interaction bipartite graph, Shearer’s bound is tight for CLLL if its base graph is a tree, and is not tight if its base graph is not a chordal graph.*

1.1.3 Critical Thresholds for Different LLLs

To determine the critical thresholds of a given interaction bipartite graph G_B is a fundamental problem, and has been extensively studied [4, 9, 13, 14, 25, 36, 39, 44, 46]. Given an interaction bipartite graph G_B , let $P_A(G_B)$, $P_V(G_B)$, $R_C(G_B)$, and $R_Q(G_B)$ be the critical thresholds for ALLL, VLLL, CLLL and QLLL, respectively. For simplicity, we may omit G_B when it is clear based on context. Here, we investigate these four kinds of critical thresholds, and particularly their relationships.

Lower bound for the gaps between critical thresholds. It has been proven that the tight bounds of VLLL and CLLL can go beyond Shearer's bound, i.e., there are gaps between the tight bounds of VLLL and CLLL and Shearer's bound. The next question is how large these gaps are. Our following theorem provides lower bounds for these gaps. Our contribution here is a general approach to study gaps quantitatively. Though we only investigate the gaps between critical thresholds here, i.e., the gaps between the tight bounds of LLLs in the direction of the *symmetric* probability vector, the techniques we provide in the proofs can be applied to other *asymmetric* directions as well.

Given a dependency graph G_D , let $\Delta(G_D)$ be the maximum degree of vertices in G_D . Given an interaction bipartite graph $G_B = ([m], [n], E)$, let $\text{Dis}(i, j)$ be the distance between i and j in $G_D(G_B)$ for any $i, j \in [m]$. Let $\Delta(G_B)$ be $\Delta(G_D(G_B))$. For a l -cyclic bipartite graph, if all the neighbors of these l left vertices have a degree of at most 2, we call the l -cyclic graph 2-discrete. With these notations, we have the following theorem.

Theorem 1.8. *Given a bipartite graph $G_B = ([m], [n], E)$ and a constant l , if for any $i \in [m]$, there is another $j \in [m]$ on a 2-discrete l_1 -cyclic graph where $\text{Dis}(i, j) \leq l - \lfloor l_1/2 \rfloor - 2$, then $P_V \geq R_C > R_Q = P_A$, $P_V - R_Q \geq \frac{1}{25} \frac{P_V^{l+3}}{(1-P_V)^l \cdot (\Delta(G_B)-1)^l}$ and $R_C - R_Q \geq \frac{1}{25} \frac{R_C^{l+3}}{(1-R_C)^l \cdot (\Delta(G_B)-1)^l}$.*

Theorem 1.8 provides the first lower bound on the gap between the critical threshold of VLLL and Shearer's bound, which constitutes a quantitative study on the difference between the classical world and the quantum world. It shows that for any finite graph (i.e., $m < +\infty$) containing a 2-discrete cyclic subgraph, P_V and R_C are exactly larger than P_A and R_Q . By Theorem 1.8, we can obtain the following corollary for cycles, which has received considerable attention in the LLL literature [24, 28].

Corollary 1.9. *For any l -cyclic graph, we have $R_C = P_V$, $R_Q = P_A$ and $P_V - P_A \geq \frac{1}{50} P_V^2 \cdot \left(\frac{P_V}{1-P_V}\right)^{\lfloor \frac{l-1}{2} \rfloor}$.*

Critical thresholds separation on lattices. Given a dependency graph G_D , it naturally defines an interaction bipartite graph $G_B(G_D)$ as follows. Regard each edge of G_D as a variable (or a qudit) and each vertex as an event (or a local Hamiltonian). An event A (or local Hamiltonian V) depends on a variable X (or a qudit \mathcal{H}) if and only if the vertex corresponding to A (or V) is an endpoint of the edge corresponding to X (or \mathcal{H}). We consider the critical thresholds of $G_B(G_D)$ for a given dependency graph G_D . Many of such graphs in the literature [4, 9, 13, 14, 25, 36, 39, 46] can be embedded into a Euclidean space naturally, and usually have a *translational unit* G_U in the sense that G_D can be viewed as the union of periodic translations of G_U . For example, a cycle of length 4 is a translational unit of the square lattice. The following is a direct application of Theorem 1.8.

Theorem 1.10. *Let G_D be a graph embedded in an Euclidean space. If G_D is a tree, then $P_A = R_Q = P_V = R_C$. Otherwise, G_D has a translational unit G_U which has a induced subgraph as a cycle, and we*

have $P_A = R_Q$, $P_V = R_C$ and $P_V - R_Q \geq \frac{1}{25} \frac{P_V^{l+3}}{(1-P_V)^l \cdot (\Delta(G_B)-1)^l}$ where l is the number of vertices of G_U .

As a concrete example, since we already know $R_Q = P_A = 0.11933888188(1)$ [46] for square lattice, by Theorem 1.10, we have $P_V = R_C \geq P_A + 2.8 \times 10^{-10} \geq 0.11933888216$. Moreover, by exploiting the specific structure of the square lattice, we can obtain a refined bound: $P_V = R_C \geq P_A + 5.943 \times 10^{-8} \geq 0.11933894131$. We calculate the lower bounds on $P_V - R_Q$ ($R_C - R_Q$) for several of the most common lattices, as summarized in Table 1, which can then be used to obtain better lower bound on P_V (R_C) exceeding R_Q directly.

Table 1: Summary of the critical threshold for various infinite interaction graphs

Lattice	P_A (R_Q)	lower bound on $P_V - R_Q$ ($R_C - R_Q$)
1-D chain	1/4 [36]	0
Triangular	$\frac{5\sqrt{5}-11}{2}$ [4, 13, 46]	6.199×10^{-8}
Square	0.1193 [14, 46]	5.943×10^{-8}
Hexagonal	0.1547 [46]	1.211×10^{-7}
Simple Cubic	0.0744 [13]	9.533×10^{-10}

Organization The organization of this paper is as follows. Section 2 provides the definitions and notations. In Section 3, we prove that Shearer’s bound is tight for QLLL. Section 4 shows that the tight region of CLLL is generally beyond Shearer’s bound. In Section 5, we investigate the critical thresholds of different LLLs and provide lower bounds for the gaps between them.

2 Definitions and Notations

Let $G_B = ([m], [n], E_B)$ be a given interaction bipartite graph and $\mathbf{r} = (r_1, r_2, \dots, r_m)$ be a given relative dimension vector. We will use boldface type, e.g., $\mathbf{r}, \mathbf{p}, \mathbf{q}, \mathbf{d}$, for vectors. For any \mathbf{r} and \mathbf{r}' of the same dimensions, we say $\mathbf{r} \geq \mathbf{r}'$ if $r_i \geq r'_i$ holds for any i . We say $\mathbf{r} > \mathbf{r}'$ if $\mathbf{r} \geq \mathbf{r}'$ and $r_i > r'_i$ holds for some i . A vector space V is the direct sum of its subspaces W_1, \dots, W_k , written as $V = W_1 \oplus W_2 \oplus \dots \oplus W_k$, if $V = W_1 + W_2 + \dots + W_k$ and the collection $\{W_1, \dots, W_k\}$ is independent. By contrast with the setting of ALLL and VLLL where probabilities can be irrational [24, 44], throughout this paper, we are only interested in finite dimensional quantum systems and restrict our attention on rational relative dimensions.

For the sake of simplicity and without loss of generality, we assume the relative dimensions are strictly positive, i.e., $\mathbf{r} \in (0, 1]^m$. Furthermore, in the whole paper except Section 3, we assume that G_B is connected (hence so is the corresponding dependency graph). In Section 3, we argue for general G_B instead, as disconnected G_B may be involved in the inductive steps.

Definition 2.1 (Hilbert Space of the Qudits). Let n be the number of qudits. Then, the Hilbert space of the quantum system is an n th-order tensor product $\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \dots \otimes \mathcal{H}_n$ over \mathbb{C} . For any $S \subseteq [m]$, let $\mathcal{H}_S := \bigotimes_{i \in S} \mathcal{H}_i$ denote the Hilbert space of the qudits in S . For example, $\mathcal{H}_{\{1,2\}} = \mathcal{H}_1 \otimes \mathcal{H}_2$. For any $j \in [n]$, let $\dim(\mathcal{H}_j)$ be the dimension of \mathcal{H}_j and $\dim(\mathcal{H}_1, \dots, \mathcal{H}_n)$ be $(\dim(\mathcal{H}_1), \dim(\mathcal{H}_2), \dots, \dim(\mathcal{H}_n))$.

We say the qudit \mathcal{H}_i is classical or a classical variable with respect to $\Pi = \sum_j \Pi_j$, if any Hamiltonians Π_j acting on it can be written as $\sum_{l \in [d_i]} |l\rangle\langle l| \otimes \Pi_{ijl}$ where $\{|l\rangle : l \in [d_i]\}$ is the computational basis of \mathcal{H}_i and Π_{ijl} is some projector in the subspace $\mathcal{H}_{[n] \setminus \{i\}}$. We will omit “with respect to $\Pi = \sum_j \Pi_j$ ” if it is clear from the context.

Definition 2.2 (Projectors, Subspaces and Relative Dimensions). Given a subspace $V \subset \mathcal{H}$, let Π_V be the projector onto V . The relative dimension of Π_V to \mathcal{H} is defined as $R_{\mathcal{H}}(\Pi_V) := \frac{\text{tr}(\Pi_V)}{\text{dim}(\mathcal{H})} = \frac{\text{dim}(V)}{\text{dim}(\mathcal{H})}$. For simplicity, we will omit “to \mathcal{H} ” and use $R(\Pi_V)$ if there is no ambiguity. It is easy to see that $R(\Pi_V)$ is a rational number. We say a set of subspaces $\mathcal{V} = \{V_1, \dots, V_m\}$ is frustration free if V_1, \dots, V_m do not span $\mathcal{H}_{[n]}$, and we will use $R(\mathcal{V})$ to represent the vector $(R(V_1), \dots, R(V_m))$. In this paper, the two terms “subspaces” and “projectors” will be used interchangeably.

Π_V is called classical if Π_V is diagonal with respect to the computational basis.

Definition 2.3 (Events and Variables). Let event set $\mathcal{A} = \{A_1, A_2, \dots, A_m\}$ be a set of events fully determined by a set of mutually independent random variables $\mathcal{X} = \{X_1, \dots, X_n\}$. Then for each $A_i \in \mathcal{A}$, there is a unique minimal subset $\mathcal{X}_i \subseteq \mathcal{X}$ that determines A_i . We denote this set of variables by $\text{vbl}(A_i)$. Let $P(\mathcal{A})$ be $(P(A_1), P(A_2), \dots, P(A_m))$. For each $X_i \in \mathcal{X}$, let U_{X_i} (U for short when it is clear from the context) stand for the universal set of the possible values of X_i .

Definition 2.4 (Neighbors). Given a bipartite graph $G_B = ([m], [n], E_B)$, let $\mathcal{N}_{G_B}(i)$ (or $\mathcal{N}(i)$ if G_B is implicit) denote the neighbors of vertex i in G_B if which side this vertex belongs to is clear from the context. We say two left vertices $i_1, i_2 \in [m]$ are neighboring or adjacent if $\mathcal{N}(i_1) \cap \mathcal{N}(i_2) \neq \emptyset$. We say a left vertex $i \in [m]$ and a right vertex $j \in [n]$ are neighboring or adjacent if $j \in \mathcal{N}(i)$.

Given a dependency graph $G_D = ([m], E_D)$ and any $i \in [m]$, let $\Gamma_i := \{j \in [m] : (i, j) \in E_D\}$ and $\Gamma_i^+ := \Gamma_i \cup \{i\}$. We say two vertices $i, j \in [m]$ are neighboring or adjacent if $j \in \Gamma_i$.

Definition 2.5 (Hamiltonians and Events on Graphs). Given a bipartite graph $G_B = ([m], [n], E_B)$, we say a set of local Hamiltonians $\mathcal{V} = \{V_1, \dots, V_m\}$ conforms with G_B , denoted by $\mathcal{V} \sim G_B$, if for any $i \in [m]$, Π_{V_i} acts trivially on qudits $[n] \setminus \mathcal{N}(i)$. Thus, we can write V_i as $V_i^{\text{loc}} \otimes \mathcal{H}_{[n] \setminus \mathcal{N}(i)}$ where $V_i^{\text{loc}} \subseteq \mathcal{H}_{\mathcal{N}(i)}$. Similarly, we can also define a set of events \mathcal{A} conforms with G_B , denoted by $\mathcal{A} \sim G_B$.

Here, we usually call G_B the interaction bipartite graph.

Definition 2.6 (Dependency Graph of G_B). Given a bipartite graph $G_B = ([m], [n], E_B)$, the corresponding dependency graph of G_B is defined as $G_D(G_B) = ([m], E_D)$, where $(i_1, i_2) \in E_D$ if and only if the left vertices i_1, i_2 are neighbors in G_B .

Given a bipartite graph G_B , we define the multivariate independence polynomial $I(G_B, \mathbf{x})$ (or $I(G_B)$ if \mathbf{x} is implicit) of G_B as that of G_D , i.e., $I(G_B, \mathbf{x}) := I(G_D(G_B), \mathbf{x})$.

Definition 2.7 (Maximum Degree). Given a dependency graph G_D , let $\Delta(G_D)$ be the maximum degree of vertices in G_D . Given an interaction bipartite graph G_B , let $\Delta(G_B)$ be $\Delta(G_D(G_B))$.

Definition 2.8 (Induced Subgraph). Given a bipartite graph $G_B = ([m], [n], E_B)$ and any $S \subseteq [m]$, let $G_B(S)$ be the induced subgraph of G_B on the left vertices in S and the right vertices in $[n]$. Given a dependency graph $G_D = ([m], E_D)$ and any $S \subseteq [m]$, let $G_D(S)$ be the induced subgraph of G_D on S .

Definition 2.9 (Cyclic Bipartite Graph). A bipartite graph $G_B = ([l], [n], E)$ is said to be l -cyclic if the base graph $G_D(G_B)$ is a cycle of length l . When $l = 3$, it additionally requires that $\bigcap_{i \in [3]} \mathcal{N}(i) = \emptyset$ holds. In case of no ambiguity, a l -cyclic bipartite graph is simply called a cyclic bipartite graph, a l -cyclic graph or a cyclic graph.

Definition 2.10 (Contained Graph). We say an interaction bipartite graph $G_B = ([m], [n], E_B)$ contains another G'_B , if there is some $S \in [m]$ such that G'_B can be obtained from $G_B(S)$ by deleting the right vertices with degree no more than one and relabeling the left vertices and the right vertices, respectively.

Intuitively, if $\mathcal{A} \sim G_B$ and G'_B is contained in G_B , then G'_B is the interaction bipartite graph for a subset of events in \mathcal{A} .

Definition 2.11 (Interior and Boundary). The classical abstract interior of the dependency graph G_D , $\mathcal{I}(G_D)$, is the set of vectors \mathbf{p} such that $\mathbb{P}(\cap_{A \in \mathcal{A}} \bar{A}) > 0$ for any event set \mathcal{A} with dependency graph G_D and probability vector \mathbf{p} . It turns out to be exactly the set of probability vectors in Shearer's bound (see Theorem 1.2). The *abstract boundary* of a graph G_D , denoted by $\partial(G_D)$, is the set $\{\mathbf{p} : (1 - \epsilon)\mathbf{p} \in \mathcal{I}(G_D) \text{ and } (1 + \epsilon)\mathbf{p} \notin \mathcal{I}(G_D) \text{ for any } \epsilon \in (0, 1)\}$. Any $\mathbf{p} \in \partial(G_D)$ is called an *abstract boundary vector* of G_D .

Similarly, we can also define $\mathcal{VI}(G_B), \mathcal{CI}(G_B), \mathcal{QI}(G_B), \mathcal{V}\partial(G_B), \mathcal{C}\partial(G_B)$ for a given bipartite graph G_B . The definitions of $\mathcal{CI}(G_B), \mathcal{QI}(G_B)$ would be a little different from the classical case, as we focus on finite dimensional quantum systems and the relative dimensions of Hamiltonians cannot be irrational (refer to Definition 4.1 for the details).

For simplify, we let $\mathcal{I}(G_B) = \mathcal{I}(G_D(G_B))$ and $\partial(G_B) = \partial(G_D(G_B))$ for a given bipartite graph G_B .

Definition 2.12 (Critical Threshold). Given a bipartite graph $G_B = ([m], [n], E_B)$, the critical threshold $P_A(G_B)$ for ALLL is the probability such that $\mathbf{p} = (P_A(G_B), P_A(G_B), \dots, P_A(G_B))$ is in $\partial(G_B)$.

Similarly, we can also define critical thresholds for VLLL (denoted by $P_V(G_B)$), QLLL (denoted by $P_Q(G_B)$) and CLLL (denoted by $P_C(G_B)$). It is easy to see that $P_A(G_B) = P_Q(G_B) \leq P_C(G_B) \leq P_V(G_B)$.

Definition 2.13 (Random Subspaces). Given an interaction bipartite graph $G_B = ([m], [n], E_B)$, an rational vector $\mathbf{r} = (r_1, \dots, r_m)$ and an integer vector $\mathbf{d} = (d_1, \dots, d_m)$, we say a subspace set \mathcal{V} of $\mathcal{H}_{[n]}$ is an instance of the setting $(G_B, \mathbf{r}, \mathbf{d})$ if $\mathcal{V} \sim G_B$, $R(\mathcal{V}) = \mathbf{r}$ and $\dim(\mathcal{H}_1, \dots, \mathcal{H}_n) = \mathbf{d}$. We further say a subspace set \mathcal{V} of $\mathcal{H}_{[n]}$ is a *random* instance of the setting $(G_B, \mathbf{r}, \mathbf{d})$ if $\dim(\mathcal{H}_1, \dots, \mathcal{H}_n) = \mathbf{d}$ and for any $i \in [m]$, $V_i = V_i^{loc} \otimes \mathcal{H}_{[n] \setminus \mathcal{N}(i)}$ where V_i^{loc} is a random subspace of $\mathcal{H}_{\mathcal{N}(i)}$ according to the Haar measure with $R(V_i^{loc}) = r_i$.

3 QLLL: Shearer's Bound Is Tight

This section aims at proving Theorem 1.3. We first present several useful tools.

3.1 Tools for QLLL

The geometrization theorem is an useful tool established by Laumann et al. [30]. With this theorem, we can show "almost all" just by showing the "existence".

Theorem 3.1 (The geometrization theorem, adapted from [30]). *Given the interaction bipartite graph $G_B = ([m], [n], E_B)$, dimension vector \mathbf{d} and relative dimensions \mathbf{r} , if there exist $\{V_i^*\}_{i \in [m]}$ of the setting $(G_B, \mathbf{r}, \mathbf{d})$ satisfying $R(\sum_{i \in [m]} V_i^*) = 1$, then for random instance \mathcal{V} of the setting $(G_B, \mathbf{r}, \mathbf{d})$, we have $R(\sum_{i \in [m]} V_i) = 1$ with probability 1.*

Another tool for depicting the tight region of QLL is the multivariate independence polynomial. Recall that for any interaction bipartite graph $G_B = ([m], [n], E_B)$ and $\mathbf{r} \in (0, 1]^m$, $I(G_B, \mathbf{r})$ is defined as $I(G_D(G_B), \mathbf{r}) = \sum_{S \in \text{Ind}(G_D(G_B))} (-1)^{|S|} \prod_{v \in S} r_v$ where $\text{Ind}(G_D(G_B))$ is the set of all independent sets of $G_D(G_B)$ by Definitions 1.1 and 2.6. For any $S \subseteq [m]$, the independence polynomial of $G_B(S)$, $I(G_B(S), \mathbf{r})$, is defined similarly by recalling that $G_B(S)$ is an induced subgraph of G_B .

Let $M(r)$ be the denominator of the fraction r in its lowest terms. By the definition of independence polynomial (i.e., Definition 1.1), it is easy to verify the following properties.

Proposition 3.2. *Given an interaction bipartite graph $G_B = ([m], [n], E_B)$ and $\mathbf{r} \in (0, 1]^m$, we have*

- (a) *If \mathbf{r} is rational, $I(G_B)$ can be written as a fraction with denominator $\prod_{i=1}^m M(r_i)$;*
- (b) *for any $i \notin S \subsetneq [m]$, $I(G_B(S \cup \{i\})) = I(G_B(S)) - r_i \cdot I(G_B(S \setminus \Gamma_i))$;*
- (c) *if $\mathcal{N}(n) = [t]$ for the right vertex n , then $I(G_B) = I(G_B([m] \setminus [t])) - \sum_{l=1}^t r_l \cdot I(G_B([m] \setminus \Gamma_l^+))$.*
- (d) *for $\emptyset \subsetneq S \subsetneq [m]$, if there is no edge between S and $[m] \setminus S$ in $G_D(G_B)$, then $I(G_B) = I(G_B(S)) \cdot I(G_B([m] \setminus S))$.*

As shown in Theorem 1.2, the independence polynomial provides an exact characterization of $\mathcal{I}(G_D)$, i.e., the tight criterion under which abstract version LLL holds. Another interesting property of the independence polynomial is as follows.

Proposition 3.3. *Given an interaction bipartite graph $G_B = ([m], [n], E_B)$ and $\mathbf{r} \in (0, 1]^m$, if $I(G_B(S), \mathbf{r}) > 0$ for any $S \subsetneq [m]$, then*

- (a) $\forall S \subseteq S' \subseteq [m]$, $I(G_B(S)) \geq I(G_B(S'))$.
- (b) $\forall S \subseteq [m]$, $I(G_B(S)) \leq 1$,
- (c) if $m \geq 2$, then $\forall i \in [m]$, $r_i < 1$.
- (d) if $I(G_B, \mathbf{r}) \leq 0$, then $G_D(G_B)$ is connected.

Proof. (a)&(b). For any $i \notin S \subsetneq [m]$, by Proposition 3.2 (b), we have $I(G_B(S \cup \{i\})) = I(G_B(S)) - r_i \cdot I(G_B(S \setminus \Gamma_i)) \leq I(G_B(S))$. In other words, $I(G_B(S))$ is non-increasing as S grows. Thus, we have for any $S \subseteq S' \subseteq [m]$, $I(G_B(S)) \geq I(G_B(S'))$. Additionally, $I(G_B(S)) \leq I(G_B(\emptyset)) = 1$ for any $S \subseteq [m]$.

(c). if $r_i = 1$, then $I(G_B(\{i\})) = 0$, a condition with the assumption that $I(G_B(S), \mathbf{r}) > 0$ for any $S \subsetneq [m]$.

(d). Suppose there exists $\emptyset \subsetneq S \subsetneq [m]$ such that there is no edge between S and $[m] \setminus S$ in $G_D(G_B)$, then by Proposition 3.2 (d), $I(G_B) = I(G_B(S))I(G_B([m] \setminus S)) \leq 0$. Hence we have either $I(G_B(S)) \leq 0$ or $I(G_B([m] \setminus S)) \leq 0$, a contradiction. □

The following proposition will also be used.

Proposition 3.4. *If there exists a subspace set \mathcal{V} of the setting $(G_B, \mathbf{r}, \mathbf{d})$ spanning the whole space, then for any \mathbf{d}' where d'_j is a multiple of d_j , there exists subspace set \mathcal{V}' of the setting $(G_B, \mathbf{r}, \mathbf{d}')$ spanning the whole space as well.*

Proof. W.l.o.g., we assume $\mathbf{d}' = (d_1, \dots, d_{n-1}, k \cdot d_n)$ where $k \geq 1$ is an integer, and $\mathcal{H}'_{[n]} = \bigotimes_{i \in [n]} \mathcal{H}'_i$ is a Hilbert space where $\dim(\mathcal{H}'_1, \dots, \mathcal{H}'_n) = \mathbf{d}'$. We decompose the qudit \mathcal{H}'_n to k orthogonal subspaces $\mathcal{H}'_n = \bigoplus_{l \in [k]} \mathcal{H}'_{nl}$ where $\dim(\mathcal{H}'_{nl}) = d_n$ for each $l \in [k]$. Thus for each l , we have $\mathcal{H}'_{[n-1]} \otimes \mathcal{H}'_{nl}$ are of dimensions \mathbf{d} , and then can be spanned by some subspace set $\mathcal{V}_l \sim G_B$ with relative dimensions \mathbf{r} (w.r.t. $\mathcal{H}'_{[n-1]} \otimes \mathcal{H}'_{nl}$). Let $\mathcal{V}' = \bigoplus_{l \in [k]} \mathcal{V}_l$, then it is easy to check that \mathcal{V}' is an instance of the setting $(G_B, \mathbf{r}, \mathbf{d}')$ and spans $\mathcal{H}'_{[n]}$. \square

3.2 Shearer's Bound Is Tight for QLLL

Theorem 3.6 shows that Shearer's bound is tight for QLLL. Shearer's bound has been shown to be a lower bound on the relative dimension of the satisfying subspace [39], more precisely,

Theorem 3.5 ([39]). *Given an interaction bipartite graph $G_B = ([m], [n], E_B)$ and rational $\mathbf{r} = (r_1, \dots, r_m) \in (0, 1]^m$, if $\mathbf{r} \in \mathcal{I}(G_B)$, then for any V_1, \dots, V_m of relative dimension r_1, \dots, r_m respectively, $1 - R(\sum_{i=1}^m V_i) \geq I(G_B, \mathbf{r})$.*

Thus it remains to show this lower bound can be achieved. Let $\mathbf{1}$ be the vector with all entries being 1.

Theorem 3.6. *Given an interaction bipartite graph $G_B = ([m], [n], E_B)$ and rational $\mathbf{r} = (r_1, \dots, r_m) \in (0, 1]^m$.*

- (a) *if $\mathbf{r} \notin \mathcal{I}(G_B)$, then there is some $\mathbf{d} = (d_1, \dots, d_n) \leq \prod_{i=1}^m M(r_i)^{2^{2^m}} \cdot \mathbf{1}$ such that for random subspace \mathcal{V} of the setting $(G_B, \mathbf{r}, \mathbf{d})$, we have $\mathbb{P}(R(\sum_{i \in [m]} V_i) = 1) = 1$.*
- (b) *Otherwise, there is some $\mathbf{d} = (d_1, \dots, d_n) \leq \prod_{i=1}^m M(r_i)^{2^{2^m+3}} \cdot \mathbf{1}$ such that for random subspace \mathcal{V} of the setting $(G_B, \mathbf{r}, \mathbf{d})$, we have $\mathbb{P}(R(\sum_{i \in [m]} V_i) = 1 - I(G_B, \mathbf{r})) = 1$.*

Recall that with the geometrization theorem, we can show “almost all” just by showing the “existence”. The proof of existence is by an inductive randomized construction. The following example is a good illustration of our main idea for the proof of Theorem 3.6 (a).

Example. The proof of Theorem 3.6 (a) is by induction on the number of left vertices in G_B . Suppose this theorem has already been verified for G_B where the number of left vertices is no more than 3. Now, we illustrate how the induction proceeds by verifying this theorem on the 4-cyclic graph (i.e. $G_B = ([4], [4], E)$ where $E = \{(i, i), (i, i+1 \pmod{4}), i \in [4]\}$) and $\mathbf{r} = (\frac{1}{3}, \frac{1}{3}, \frac{1}{4}, \frac{1}{4})$. Note that the base graph of G_B is a cycle of length 4, hence $I(G_B, \mathbf{r}) = 1 - \sum_{i=1}^4 r_i + r_1 \cdot r_3 + r_2 \cdot r_4 = 0$. Our construction here is randomized: Let $\mathcal{H}_2 := e_1 \oplus e_2 \simeq \mathbb{C}^2$, and

1. $V_1^{loc} = V'_1 \otimes e_1$ where V'_1 is a random subspace of \mathcal{H}_1 with $R(V'_1) = \frac{2}{3}$,
2. $V_2^{loc} = e_2 \otimes V'_2$ where V'_2 is a random subspace of \mathcal{H}_3 with $R(V'_2) = \frac{2}{3}$,
3. V_3^{loc} is a random subspace of $\mathcal{H}_3 \otimes \mathcal{H}_4$ with $R(V_3) = \frac{1}{4}$,
4. V_4^{loc} is a random subspace of $\mathcal{H}_4 \otimes \mathcal{H}_1$ with $R(V_4) = \frac{1}{4}$,

where the dimensions of $\mathcal{H}_1, \mathcal{H}_3, \mathcal{H}_4$, namely (d_1, d_3, d_4) , will be determined later.

Consider the subspace $e_1 \otimes \mathcal{H}_{1,3,4} := \mathcal{H}'$ and the associated Hamiltonians $V_1 \cap \mathcal{H}', V_3 \cap \mathcal{H}', V_4 \cap \mathcal{H}'$, whose relative dimensions to \mathcal{H}' are $R(V_1') = \frac{2}{3}, r_3 = \frac{1}{4}$ and $r_4 = \frac{1}{4}$ respectively. Note that the base graph of this subsystem is a 3-path, and the independence polynomial becomes $1 - R(V_1') - r_3 - r_4 + R(V_1') \cdot r_3 = 0$. Then by the induction hypothesis, there is some d_1', d_3' and d_4' such that $V_1 \cap \mathcal{H}', V_3 \cap \mathcal{H}', V_4 \cap \mathcal{H}'$ spans \mathcal{H}' w.p.1; Similarly, it also holds that $e_2 \otimes \mathcal{H}_{1,3,4}$ is included in $V_2 + V_3 + V_4$ w.p.1 for some d_1'', d_3'' and d_4'' . Therefore, due to Proposition 3.4 and Theorem 3.1, for (d_1, d_3, d_4) which is common multiple of (d_1', d_3', d_4') and (d_1'', d_3'', d_4'') , we have $\{V_i\}_{i=1}^4$ span the whole space w.p.1 by the union bound. Now we have shown the existence of such a subspace. By geometrization theorem, Theorem 3.6 (a) for the 4-cyclic graph and $\mathbf{r} = (\frac{1}{3}, \frac{1}{3}, \frac{1}{4}, \frac{1}{4})$ becomes immediate.

Now we are ready to prove Theorem 3.6.

Proof. *Part (a).* The proof is by induction on the number of left vertices in G_B .

Basic: If the number of left vertices in G_B is no more than 1, the theorem holds trivially.

Induction: We assume this theorem holds for any interaction bipartite graph where the number of left vertices is no more than $m - 1$. In the following, we prove that the theorem also holds for any graph G_B where the number of left vertices is m .

If $\mathbf{r} \notin \mathcal{I}(G_B)$, we have $\exists S \subseteq [m]$ s.t. $I(G_B(S), \mathbf{r}) \leq 0$. Let $S \subseteq [m]$ be of the minimal size such that $I(G_B(S), \mathbf{r}) \leq 0$, so we have that $\mathbf{r} \notin \mathcal{I}(G_B(S))$. If $S \subsetneq [m]$, then by the induction hypothesis, there is some $\mathbf{d} \leq \prod_{i \in S} M(r_i)^{2^{2^m}} \cdot \mathbf{1}$ such that there exists an instance \mathcal{V} of the setting $(G_B(S), \mathbf{r}, \mathbf{d})$ satisfying $R(\sum_{i \in S} V_i) = 1$. Let $\mathbf{d}' = \prod_{i \notin S} M(r_i)^{2^{2^m}} \cdot \mathbf{d} \leq \prod_{i=1}^m M(r_i)^{2^{2^m}} \cdot \mathbf{1}$, due to Proposition 3.4, there exists \mathcal{V}' of the setting $(G_B(S), \mathbf{r}, \mathbf{d}')$ spanning the whole space. For any $i \notin S$, note that each $\dim(\mathcal{H}_j)$ where $j \in [n]$ is a multiple of $M(r_i)$, so we can select an arbitrary subspace of $\mathcal{H}_{\mathcal{N}(i)}$ with relative dimension r_i as V_i^{loc} , and obtain an instance of the setting $(G_B, \mathbf{r}, \mathbf{d}')$ which spans the whole space as well. Finally, the theorem is immediate by Theorem 3.1.

In the following we assume $S = [m]$, thus $I(G_B(T), \mathbf{r}) > 0$ for any $T \subsetneq [m]$ by our assumption $S = [m]$ is of the minimal size such that $I(G_B(S), \mathbf{r}) \leq 0$. By Proposition 3.3 (d), $G_D(G_B)$ is connected. Thus, there must be a right vertex in G_B with at least two neighbors in the left vertices. Without loss of generality, we assume the right vertex n is such a vertex and $\mathcal{N}(n) = [t]$ where $t \geq 2$.

By the geometrization theorem, we only need to show the existence of V_1, \dots, V_m with given relative dimension \mathbf{r} spanning the whole space. The construction of V_1, \dots, V_m is as follows. We decompose \mathcal{H}_n into t orthogonal subspaces $\mathcal{H}_n^1, \dots, \mathcal{H}_n^t$ arbitrarily where

$$\dim(\mathcal{H}_n^i) = \frac{r_i \cdot I(G_B([m] \setminus \Gamma_i^+))}{\sum_{l=1}^t r_l \cdot I(G_B([m] \setminus \Gamma_l^+))} \cdot \dim(\mathcal{H}_n) \quad (1)$$

for any $i \in [t]$. To show that this is a reasonable decomposition, we need to guarantee the following: the denominator is not 0, the sum of dimensions of the orthogonal subspaces is exactly the dimension of whole space, and the dimension of every \mathcal{H}_n^i are positive integers by choosing an appropriate $\dim(\mathcal{H}_n)$. All of these are true, as

1. $\forall i \in [t], r_i \cdot I(G_B([m] \setminus \Gamma_i^+)) > 0$, by the assumption that $r_i > 0$ and $I(G_B(T), \mathbf{r}) > 0$ for any $T \subsetneq [m]$.
2. It is immediate that $\sum_{l=1}^t \dim(\mathcal{H}_n^l) = \dim(\mathcal{H}_n)$ by the definition of \mathcal{H}_n^i .

3. $\forall i \in [t]$, by applying Proposition 3.2 (a) on $G_B([m] \setminus \Gamma_i^+)$, it is easy to see that $r_i \cdot I(G_B([m] \setminus \Gamma_i^+))$ can be written as a fraction with denominator $\prod_{i \in [m] \setminus \Gamma_i} M(r_i)$. Moreover, we have the following claim:

Claim. $\sum_{l=1}^t r_l \cdot I(G_B([m] \setminus \Gamma_l^+)) \leq 2$.

Proof. As Proposition 3.2 (c) says, $\sum_{l=1}^t r_l \cdot I(G_B([m] \setminus \Gamma_l^+)) = I(G_B([m] \setminus [t])) - I(G_B)$. First, due to Proposition 3.3 (b), we have $I(G_B([m] \setminus [t])) \leq 1$. Moreover, by applying Proposition 3.2 (b), $I(G_B) = I(G_B([m] \setminus \{1\})) - r_1 \cdot I(G_B([m] \setminus \Gamma_1^+)) \geq 0 - 1 \cdot 1 \geq -1$, which implies the conclusion immediately. \square

Therefore, there exists a $d_n \leq 2 \prod_{i=1}^m M(r_i) \leq \prod_{i=1}^m M(r_i)^{2^{2m}}$ to make all $\dim(\mathcal{H}_n^i)$ positive integers. Here the last inequality is because due to Proposition 3.3 (c), $r_i < 1$, thus $M(r_i) \geq 2$.

Therefore, we can choose V_1, \dots, V_m with relative dimensions \mathbf{r} to $\mathcal{H}_{[n]}$ randomly as follows:

- For $i \leq t$, if $r_i \cdot \frac{\dim(\mathcal{H}_n)}{\dim(\mathcal{H}_n^i)} \leq 1$, let V_i^{loc} be a random subspace of $\mathcal{H}_{\mathcal{N}(i) \setminus \{n\}} \otimes \mathcal{H}_n^i$ with relative dimension $r_i \cdot \frac{\dim(\mathcal{H}_n)}{\dim(\mathcal{H}_n^i)}$ to $\mathcal{H}_{\mathcal{N}(i) \setminus \{n\}} \otimes \mathcal{H}_n^i$; Otherwise, let V_i^{loc} be $(\mathcal{H}_{\mathcal{N}(i) \setminus \{n\}} \otimes \mathcal{H}_n^i) \oplus (V_i^{loc, of} \otimes (\bigoplus_{l \neq i} \mathcal{H}_n^l))$, where $V_i^{loc, of}$ is an arbitrarily subspace of $\mathcal{H}_{\mathcal{N}(i) \setminus \{n\}}$ of relative dimension (w.r.t. $\mathcal{H}_{\mathcal{N}(i) \setminus \{n\}}$) $(r_i \cdot \frac{\dim(\mathcal{H}_n)}{\dim(\mathcal{H}_n^i)} - 1) \cdot \frac{\dim(\mathcal{H}_n)}{\sum_{l \neq i} \dim(\mathcal{H}_n^l)}$.
- For $i > t$, let V_i^{loc} be a random subspace of $\mathcal{H}_{\mathcal{N}(i)}$ with relative dimension r_i to $\mathcal{H}_{\mathcal{N}(i)}$,

where the dimensions of $\mathcal{H}_1, \dots, \mathcal{H}_{n-1}$, namely (d_1, \dots, d_{n-1}) , will be determined later.

Given $i \in [t]$, consider the subspace $\mathcal{H}_{[n-1]} \otimes \mathcal{H}_n^i$ and the associated Hamiltonians V_i, V_{t+1}, \dots, V_m restricted on this subspace. On the one hand, if $r_i \cdot \frac{\dim(\mathcal{H}_n)}{\dim(\mathcal{H}_n^i)} \leq 1$, the relative dimension of V_i in this subspace becomes $r_i \cdot \frac{\dim(\mathcal{H}_n)}{\dim(\mathcal{H}_n^i)}$, while those of V_{t+1}, \dots, V_m remain the same. Thus, by Proposition 3.2 (b), the independence polynomial of the subsystem turns out to be $I(G_B([m] \setminus [t])) - r_i \cdot \frac{\dim(\mathcal{H}_n)}{\dim(\mathcal{H}_n^i)} \cdot I(G_B([m] \setminus \Gamma_i^+))$. By Equation (1), we have

$$I(G_B([m] \setminus [t])) - r_i \cdot \frac{\dim(\mathcal{H}_n)}{\dim(\mathcal{H}_n^i)} \cdot I(G_B([m] \setminus \Gamma_i^+)) = I(G_B([m] \setminus [t])) - \sum_{l=1}^t r_l \cdot I(G_B([m] \setminus \Gamma_l^+)).$$

By Proposition 3.2 (c), we have

$$I(G_B([m] \setminus [t])) - \sum_{l=1}^t r_l \cdot I(G_B([m] \setminus \Gamma_l^+)) = I(G_B).$$

By our assumptions $S = [m]$ and $I(G_B(S), \mathbf{r}) \leq 0$, we have $I(G_B) \leq 0$. Combined with above two equalities, we have

$$I(G_B([m] \setminus [t])) - r_i \cdot \frac{\dim(\mathcal{H}_n)}{\dim(\mathcal{H}_n^i)} \cdot I(G_B([m] \setminus \Gamma_i^+)) = I(G_B) \leq 0.$$

Thus, by the induction hypothesis, we have for any $i \in [t]$, V_i, V_{t+1}, \dots, V_m span the whole subspace $\mathcal{H}_{[n-1]} \otimes \mathcal{H}_n^i$ with probability 1 for some $(d_1^{(i)}, \dots, d_{n-1}^{(i)}) \leq (M(r_i \cdot \frac{\dim(\mathcal{H}_n)}{\dim(\mathcal{H}_n^i)}) \cdot \prod_{i=t+1}^m M(r_i))^{2^{(m-t+1)}} \cdot \mathbf{1} \leq \prod_{i=1}^m M(r_i)^{2^{2(m-t+1)} \cdot 2} \cdot \mathbf{1}$. The last inequality is due to the easy observation that $M(r_i \cdot \frac{\dim(\mathcal{H}_n)}{\dim(\mathcal{H}_n^i)}) \leq \prod_{i=1}^m M(r_i)$.

On the other hand, if $r_i \cdot \frac{\dim(\mathcal{H}_n)}{\dim(\mathcal{H}_n^i)} > 1$, by the definition of V_i^{loc} , we have $\mathcal{H}_{[n-1]} \otimes \mathcal{H}_n^i \subseteq V_i$. In addition, it is easy to see the denominator of relative dimension of $V_i^{loc,of}$ to $\mathcal{H}_{\mathcal{N} \setminus \{n\}}$ in its lowest terms is no more than $\prod_{i=1}^m M(r_i)^2$. So there is $(d_1^{(i)}, \dots, d_{n-1}^{(i)}) \leq \prod_{i=1}^m M(r_i)^3 \cdot \mathbf{1} \leq \prod_{i=1}^m M(r_i)^{2^{2(m-t+1)}} \cdot \mathbf{1}$ such that the dimensions of V_i^{loc} and V_{t+1}, \dots, V_m are positive integers. Meanwhile, it is easy to verify that for any $i \in [t]$, V_i, V_{t+1}, \dots, V_m span the whole subspace $\mathcal{H}_{[n-1]} \otimes \mathcal{H}_n^i$ with probability 1.

For any $j \in [n-1]$, let d_j be the least common multiple of $d_1^{(1)}, \dots, d_{n-1}^{(t)}$. We have $(d_1, \dots, d_{n-1}) \leq \prod_{i=1}^m M(r_i)^{2^{2(m-t+1)}} \cdot \mathbf{1} \leq \prod_{i=1}^m M(r_i)^{2^{2m}} \cdot \mathbf{1}$ as $t \geq 2$. Due to Proposition 3.4 and Theorem 3.1, for such (d_1, \dots, d_{n-1}) , we have V_1, \dots, V_m span the whole space $\mathcal{H}_{[n]}$ with probability 1 by the union bound and finish the proof.

Part(b). Given an interaction bipartite graph $G_B = ([m], [n], E_B)$ and rational $\mathbf{r} = (r_1, \dots, r_m)$ where $\mathbf{r} \in \mathcal{I}(G_B)$, we construct another G'_B and \mathbf{r}' as follows. Let $G'_B = ([m+1], [n], E'_B)$ where $E'_B = E_B \cup \{(m+1, 1), (m+1, 2), \dots, (m+1, n)\}$, i.e., $\mathcal{N}(m+1) = [n]$. Let \mathbf{r}' be $(r_1, \dots, r_m, r_{m+1})$ where $r_{m+1} = I(G_B, \mathbf{r})$. Then the independence polynomial of G'_B is $I(G'_B, \mathbf{r}') = I(G_B, \mathbf{r}) - r_{m+1} = 0$.

It is easy to see $M(r_{m+1}) \leq \prod_{i=1}^m M(r_i)$. Applying Part (a) to G'_B and \mathbf{r}' , we have there is some $\mathbf{d} = (d_1, \dots, d_n) \leq \prod_{i=1}^{m+1} M(r_i)^{2^{2(m+1)}} \cdot \mathbf{1} \leq \prod_{i=1}^m M(r_i)^{2^{2m+3}} \cdot \mathbf{1}$ such that for random subspace \mathcal{V}' of the setting $(G'_B, \mathbf{r}', \mathbf{d})$, $\mathbb{P}(R(\sum_{i \in [m+1]} V'_i) = 1) = 1$. Thus, we have for random subspace $\mathcal{V} = \{V'_1, \dots, V'_m\}$ of the setting $(G_B, \mathbf{r}, \mathbf{d})$, $\mathbb{P}(R(\sum_{i \in [m]} V'_i) \geq 1 - I(G_B, \mathbf{r})) = 1$, which follows from the property $R(A+B) \leq R(A) + R(B)$ [3]. By Theorem 3.5, we also have $R(\sum_{i \in [m]} V'_i) \leq 1 - I(G_B, \mathbf{r})$. Then, the conclusion is immediate. \square

In the following, we prove Theorem 1.3, which extends Theorem 3.6 to all large enough qudits.

Theorem 1.3 (Restated). *For any interaction bipartite graph $G_B = ([m], [n], E_B)$ and any $\epsilon > 0$, there is $\mathbf{D} = (D_1, \dots, D_n) \leq \lceil 2 \lceil 2m/\epsilon \rceil^{m2^{2m}} mn/\epsilon \rceil \cdot \mathbf{1}$, such that for any $\mathbf{d} \geq \mathbf{D}$, any rational $\mathbf{r} \in (0, 1]^m$ and any rational $\mathbf{r}' \geq \mathbf{r} + \frac{\epsilon}{m} \cdot \mathbf{1}$, we have*

(a) *If \mathbf{r} is beyond Shearer's bound, then almost all \mathcal{V}' of the setting $(G_B, \mathbf{r}', \mathbf{d})$ satisfies $R(\sum_{i=1}^m V'_i) = 1$, thus $R(\sum_{i=1}^m V_i) \in [1 - \epsilon, 1]$ for almost all \mathcal{V} of the setting $(G_B, \mathbf{r}, \mathbf{d})$.*

(b) *Otherwise, almost all \mathcal{V}' of the setting $(G_B, \mathbf{r}', \mathbf{d})$ satisfies $R(\sum_{i=1}^m V'_i) \geq 1 - I(G_B, \mathbf{r})$, thus $R(\sum_{i=1}^m V_i) \in [1 - I(G_B, \mathbf{r}) - \epsilon, 1 - I(G_B, \mathbf{r})]$ for almost all \mathcal{V} of the setting $(G_B, \mathbf{r}, \mathbf{d})$.*

Proof. Let $\mathcal{R} = \{\frac{1}{\lceil 2m/\epsilon \rceil} (z_1, \dots, z_m) : \forall i \in [m], z_i \in [\lceil 2m/\epsilon \rceil - 1] \cup \{1\}\}$ be a finite set of rational vectors, then for any $\mathbf{r} \in (0, 1]^m$, there is some rational $\mathbf{r}^* \in \mathcal{R}$ such that $\mathbf{r} \leq \mathbf{r}^* \leq \mathbf{r} + \frac{\epsilon}{2m} \cdot \mathbf{1}$. According to Theorem 3.6, for any $\mathbf{r}^* \in \mathcal{R}$, there is some $\mathbf{d}^* \leq \prod_{i=1}^m M(r_i^*)^{2^{2m}} \leq \lceil 2m/\epsilon \rceil^{m2^{2m}}$ such that the setting $(G_B, \mathbf{r}^*, \mathbf{d}^*)$ admits an instance \mathcal{V}^* satisfying:

1. If \mathbf{r}^* is beyond Shearer's bound, then \mathcal{V}^* spans the whole space;

2. Otherwise, $R(\sum_{i \in [m]} V_i^*) = 1 - I(G_B, \mathbf{r}^*)$.

We set D_j as $\max_{\mathbf{r}^* \in \mathcal{R}} \lceil 2d_j^* mn/\epsilon \rceil$ for each $j \in [n]$. It is immediate that $D_j \leq \lceil 2 \lceil 2m/\epsilon \rceil^{m2^{2m}} mn/\epsilon \rceil$.

Part (a). Suppose \mathbf{r} is beyond Shearer's bound and \mathbf{r}^* satisfies $\mathbf{r} \leq \mathbf{r}^* \leq \mathbf{r} + \frac{\epsilon}{2m} \cdot \mathbf{1}$. For any $\mathbf{d} \geq \mathbf{D}$ and any $\mathbf{r}' \geq \mathbf{r}^* + \frac{\epsilon}{2m} \cdot \mathbf{1}$, we construct a subspace set $\mathcal{V}' \sim G_B$ with $\dim(\mathcal{V}') \leq \mathbf{r}'$ spanning the whole space $\mathcal{H}_{[n]}$ where $\dim(\mathcal{H}_1, \dots, \mathcal{H}_n) = \mathbf{d}$. Then the conclusion is immediate by the geometrization theorem. The construction of \mathcal{V}' is as follows.

1. $\forall j \in [n]$, decompose \mathcal{H}_j into two orthogonal subspaces $\mathcal{H}_j = \mathcal{H}_j^a \oplus \mathcal{H}_j^b$ with $\dim(\mathcal{H}_j^a) = \lfloor \frac{d_j}{d_j^*} \rfloor \cdot d_j^*$ and $\dim(\mathcal{H}_j^b) = d_j \pmod{d_j^*}$. Note that $\dim(\mathcal{H}_j^a)$ is a multiple of d_j^* . Thus, by Proposition 3.4, there is a subspace set $\mathcal{V}^a \sim G_B$ spanning $\mathcal{H}_{[n]}^a$ with relative dimensions \mathbf{r}^* to $\mathcal{H}_{[n]}^a$.
2. For any $i \in [m]$, let $V_i' := V_i^a \oplus (\sum_{j \in \mathcal{N}(i)} \mathcal{H}_j^b \otimes \mathcal{H}_{[n] \setminus \{j\}})$.

In the following, we only need to verify that \mathcal{V}' spans the whole space and $R(\mathcal{V}') \leq \mathbf{r}^* + \frac{\epsilon}{2m} \cdot \mathbf{1}$. Note that each $\mathcal{H}_j^b \otimes \mathcal{H}_{[n] \setminus \{j\}} \subseteq V_i'$ where $i \in \mathcal{N}(j)$ and $\sum_{i \in [m]} V_i^a = \mathcal{H}_{[n]}^a$, hence \mathcal{V}' spans the whole space. Meanwhile, for each $j \in [n]$, we have $\dim(\mathcal{H}_j^b) \leq d_j^*$. Thus, for each $i \in [m]$, we have $R(V_i') \leq R(V_i^a) + \sum_{j \in \mathcal{N}(i)} \frac{d_j^*}{d_j} \leq r_i^* + n \cdot \frac{\epsilon}{2mn} \leq r_i^* + \frac{\epsilon}{2m}$.

Part (b). Suppose \mathbf{r} is in Shearer's bound and \mathbf{r}^* satisfying $\mathbf{r} \leq \mathbf{r}^* \leq \mathbf{r} + \frac{\epsilon}{2m} \cdot \mathbf{1}$. For any $\mathbf{d} \geq \mathbf{D}$ and any $\mathbf{r}' \geq \mathbf{r}^* + \frac{\epsilon}{2m} \cdot \mathbf{1}$, we claim that almost all such subspace set \mathcal{V}' of the setting $(G_B, \mathbf{r}', \mathbf{d})$ satisfying $R(\sum_{i \in [m]} V_i') \geq 1 - I(G_B, \mathbf{r}^*)^+ \geq 1 - I(G_B, \mathbf{r})$, which concludes the proof. Here, we define $I(G_B, \mathbf{r}^*)^+$ as $I(G_B, \mathbf{r}^*)$ if \mathbf{r}^* is in Shearer's bound, as 0 otherwise.

Define G_B' as in the proof of Part (b) in Theorem 3.6 and let $\mathbf{r}'' = (\mathbf{r}', I(G_B, \mathbf{r}^*)^+)$. Then by the geometrization theorem, it is sufficient to show there exist an instance $\mathcal{V}'' \cup \{V_{m+1}''\}$ of the setting $(G_B', \leq \mathbf{r}'', \mathbf{d})$ spanning the whole space $\mathcal{H}_{[n]}$. The construction is similar to that of Part (a).

1. Define \mathcal{H}_j^a and \mathcal{H}_j^b as in Part (a). Again, since $\dim(\mathcal{H}_j^a)$ is a multiple of d_j^* , by Proposition 3.4, there is a subspace set $\mathcal{V}^a \cup \{V_{m+1}^a\} \sim G_B'$ spanning $\mathcal{H}_{[n]}^a$ with relative dimensions $(\mathbf{r}^*, I(G_B, \mathbf{r}^*)^+)$ to $\mathcal{H}_{[n]}^a$.
2. Let $V_i'' := V_i^a \oplus (\sum_{j \in \mathcal{N}(i)} \mathcal{H}_j^b \otimes \mathcal{H}_{[n] \setminus \{j\}})$ for any $i \in [m]$, and $V_{m+1}'' = V_{m+1}^a$.

Similar to the analysis in Part (a), we can check that $\mathcal{V}'' \cup \{V_{m+1}''\}$ spans $\mathcal{H}_{[n]}$, $R(V_i'') \leq r_i^* + \frac{\epsilon}{2m}$ for $i \leq m$, and $R(V_{m+1}'') \leq I(G_B, \mathbf{r}^*)^+$. \square

4 CLLL: Beyond Shearer's Bound

In this section, we focus on the tight region of commuting LLL. Firstly, we give the definitions of interior, boundary and gap.

Additional to the properties for general subspaces proved in [3], we prove some additional properties of the *relative dimension* only holding for the commuting case. These additional properties will be used in the following definitions and proofs implicitly.

Lemma 4.1. *For any commuting subspaces V, W, V_1, \dots, V_n , the following hold*

- (i) *Mutual independence for orthogonal complementary space: let V^\perp be the orthogonal complement of V , then $R(V|W) + R(V^\perp|W) = 1$ (refer to the table on Page 3 for the definition of $R(A|B)$). Thus if $R(V \cap W) = R(V) \cdot R(W)$, then $R(V^\perp \cap W) = R(V^\perp) \cdot R(W)$;*
- (ii) *Inclusion-exclusion principle:*

$$R\left(\sum_{i=1}^n V_i\right) = \sum_{k=1}^n (-1)^{k+1} \left(\sum_{1 \leq i_1 < \dots < i_k \leq n} R(V_{i_1} \cap \dots \cap V_{i_k}) \right).$$

Proof. Because subspaces V, W and V_1, V_2, \dots, V_n commute, all the terms can be diagonalized simultaneously with respect to an orthonormal basis, denoted by $\{|e_1\rangle, \dots, |e_t\rangle\}$. Then we can define a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ as follows. Let $\Omega = \{|e_1\rangle, \dots, |e_t\rangle\}$, $\mathbb{P}(|e_i\rangle) = 1/t$ for each i and $\mathcal{F} = 2^\Omega$. Let $A_V = \{|e_i\rangle : |e_i\rangle \in V\}$, $A_W = \{|e_i\rangle : |e_i\rangle \in W\}$, then $R(V) = \mathbb{P}(A_V)$ and $R(W) = \mathbb{P}(A_W)$. Thus, we have $A_{V \cap W} = A_V \cap A_W$ and $A_{V+W} = A_V \cup A_W$. It is easy to see the first property holds according to the analogue property of probability.

Define A_{V_1}, \dots, A_{V_n} similarly. It is also easy to see the second property holds. \square

Definition 4.1 (Commuting Interior). The *commuting interior* of an interaction bipartite graph $G_B = ([m], [n], E_B)$, denoted by $\mathcal{CI}(G_B)$, is the set $\{\mathbf{r} \in (0, 1)^m$: there is a rational vector $\mathbf{r}' \geq \mathbf{r}$ such that $R(\sum_{V \in \mathcal{V}} V) < 1$ for any commuting subspace set $\mathcal{V} \sim G_B$ with $R(\mathcal{V}) = \mathbf{r}'\}$. The definition of $\mathcal{QI}(G_B)$ is the same except that the subspaces in \mathcal{V} need not to commute.

The following two conclusions, Lemma 4.2 and Corollary 4.3, are to show the above definition of *Commuting Interior* is reasonable.

Lemma 4.2 (Monotonicity Lemma). *Suppose there is a commuting subspace set $\mathcal{V} \sim G_B$ with $R(\mathcal{V}) = \mathbf{r}$ such that $R(\sum_{V \in \mathcal{V}} V) = 1$. Then for any rational relative dimension vector $\mathbf{r}' \geq \mathbf{r}$, there is a commuting subspace set $\mathcal{V}' \sim G_B$ with $R(\mathcal{V}') = \mathbf{r}'$ such that $R(\sum_{V' \in \mathcal{V}'} V') = 1$.*

The monotonicity is obvious for VLLL and QLLL, and becomes less trivial for CLLL due to the commutation restriction. Here, we get around this problem by adding new qudits.

Proof. Without loss of generality, we assume that $\mathbf{r}' = (r_1 + \epsilon, r_2, \dots, r_m)$ and V_1 acts on \mathcal{H}_1 . Since \mathbf{r}' and \mathbf{r} are both rational vectors, $\frac{\epsilon}{1-r_1}$ is rational as well. Suppose $\frac{\epsilon}{1-r_1} = \frac{a}{b}$ where a and b are integers. Let $\mathcal{H}'_1 = \mathcal{H}_1 \otimes \mathcal{H}_1^c$ where $\dim(\mathcal{H}_1^c) = b$, and $\mathcal{H}'_i = \mathcal{H}_i$ for any $i \geq 2$. Thus the whole vector space is $\bigotimes_{i=1}^n \mathcal{H}'_i = \bigotimes_{i=1}^n \mathcal{H}_i \otimes \mathcal{H}_1^c$.

We construct the subspace set \mathcal{V}' as follows. Let $V'_1 = (\mathcal{H}_1^c \otimes V_1) + (W \otimes \bigotimes_{i=1}^n \mathcal{H}_i)$ where W can be any subspace of \mathcal{H}_1^c with dimension a . For each $i \geq 2$, let $V'_i = V_i \otimes \mathcal{H}_1^c$. It is not difficult to verify that \mathcal{V}' satisfies the conditions in the Lemma. \square

The following is an immediate corollary of Lemma 4.2.

Corollary 4.3. *Given an interaction bipartite graph $G_B = ([m], [n], E_B)$ and a rational $\mathbf{r} \in \mathcal{CI}(G_B)$, $R(\sum_{V \in \mathcal{V}} V) < 1$ holds for any commuting subspace set $\mathcal{V} \sim G_B$ with $R(\mathcal{V}) = \mathbf{r}$.*

Thus, $\mathcal{CI}(G_B)$ consists of two sets: one is the set of *rational* vectors \mathbf{r} such that $R(\sum_{V \in \mathcal{V}} V) < 1$ for any commuting subspace set $\mathcal{V} \sim G_B$ with $R(\mathcal{V}) = \mathbf{r}$, and the other is the set of *irrational* vectors to make $\mathcal{CI}(G_B)$ continuous.

Definition 4.2 (Commuting Boundary). The *commuting boundary* of an interaction bipartite graph $G_B = ([m], [n], E_B)$, denoted by $\mathcal{C}\partial(G_B)$, is the set of vectors \mathbf{r} in $(0, 1]^m$ such that $(1 - \epsilon)\mathbf{r} \in \mathcal{CI}(G_B)$ and $(1 + \epsilon)\mathbf{r} \notin \mathcal{CI}(G_B)$ for any $\epsilon \in (0, 1)$. Any $\mathbf{r} \in \mathcal{C}\partial(G_B)$ is called a *commuting boundary vector* of G_B .

According to the definition, the following proposition is obvious.

Proposition 4.4. *Given an interaction bipartite graph $G_B = ([m], [n], E_B)$, for any $\mathbf{r} \in (0, 1]^m$, there exists a unique $\lambda > 0$ such that $\lambda\mathbf{r} \in \mathcal{C}\partial(G_B)$.*

Similar to the classical case [24], the idea of exclusiveness is the key of many proofs for CLLL.

Definition 4.3 (Commuting Exclusiveness). A commuting subspace set \mathcal{V} is called *exclusive* with respect to an interaction bipartite graph G_B , if $\mathcal{V} \sim G_B$ and $R(V_i \cap V_j) = 0$ (or $V_i \perp V_j$) for any i, j where $i \in \Gamma_j$. We do not mention “with respect to G_B ” if it is clear from the context. An exclusive subspace set commute implicitly.

Recall that $\partial(G_B)$ is abstract boundary of the interaction bipartite graph and $\mathcal{I}(G_B) := \mathcal{I}(G_D(G_B))$ is the classical abstract interior (see Definition 2.11). By the definition of Shearer’s bound (i.e., Definition 1.2) and Theorem 1.2, it is easy to check that $\mathcal{I}(G_B)$ is an open set, i.e., $\mathcal{I}(G_B) \cap \partial(G_B) = \emptyset$.

It is easy to see that $\mathcal{I}(G_B) \subseteq \mathcal{CI}(G_B)$ for any interaction bipartite graph G_B . Here, we care about whether the boundaries $\partial(G_B)$ and $\mathcal{C}\partial(G_B)$ are same.

Definition 4.4 (Gap). An interaction bipartite graph G_B is called *gapful for CLLL in the direction of $\mathbf{r} \in (0, 1]^m$* , if there is a gap between $\partial(G_B)$ and $\mathcal{C}\partial(G_B)$ in this direction, i.e., a $\lambda > 0$ such that $\lambda \mathbf{r} \in (\mathcal{CI}(G_B) \cup \mathcal{C}\partial(G_B)) \setminus (\mathcal{I}(G_B) \cup \partial(G_B))$, otherwise it is called *gapless in this direction*. G_B is said to be *gapful for CLLL* if it is gapful in some direction, otherwise it is *gapless*. Similarly, we can also define *gapful/gapless* for VLLL. We do not mention “for CLLL” or “for VLLL” if it is clear from context.

Remark. Another natural definition of gapful/gapless is as follows: an interaction bipartite graph G_B is called *gapful for CLLL in the direction of $\mathbf{r} \in (0, 1]^m$* , if there is a $\lambda > 0$ such that $\lambda \mathbf{r} \in \mathcal{CI}(G_B) \setminus \mathcal{I}(G_B)$, otherwise it is called *gapless in this direction*. The only difference of these two definitions appears in the case there is a $\lambda_0 > 0$ such that $\lambda_0 \mathbf{r} \in \mathcal{C}\partial(G_B) \cap \mathcal{CI}(G_B)$ but $\lambda_0 \mathbf{r} \in \partial(G_B)$, $\lambda_0 \mathbf{r} \notin \mathcal{I}(G_B)$. Informally, the boundaries in direction \mathbf{r} are the same, but the interiors are different. We use Definition 4.4 because this case should be regarded as gapless since the boundaries are the same.

4.1 Tools for CLLL

In this subsection, we present several useful tools for depicting the tight region of CLLL, including a sufficient condition for the equivalence of CLLL and VLLL, a sufficient and necessary condition for deciding whether Shearer’s bound is tight and the reduction rules.

4.1.1 Solitary Qudits Are Classical

In this subsection, we prove Theorem 4.8, which enables us to generalize many useful tools from VLLL to CLLL.

Definition 4.5. Given an interaction bipartite graph $G_B = ([m], [n], E_B)$, we say a right vertex $j \in [n]$ is *solitary* if for any $i_1, i_2 \in \mathcal{N}(j)$ we have $\mathcal{N}(i_1) \cap \mathcal{N}(i_2) = \{j\}$.

Lemma 4.5. *Given an interaction bipartite graph $G_B = ([m], [n], E_B)$, if there exists commuting $\mathcal{V} \sim G_B$ spanning the whole space, then there exists another commuting $\mathcal{V}' \sim G_B$ where $R(\mathcal{V}') = R(\mathcal{V})$ spanning the whole space and satisfying that for each solitary $j \in [n]$, \mathcal{H}_j is classical.*

The idea of Lemma 4.5 is to dissect the structure of commuting local Hamiltonians by using Bravyi and Vyalii’s *Structure Lemma* [7].

Lemma 4.6 (Structure Lemma, adapted from [7]). *Suppose $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ are complex Euclidean spaces, Π_V and Π_W are projection operators acting on $\mathcal{X} \otimes \mathcal{Y}$ and $\mathcal{Y} \otimes \mathcal{Z}$ respectively. If $[\Pi_V, \Pi_W] = 0$, then \mathcal{Y} can be decomposed to some orthogonal subspaces $\mathcal{Y} = \bigoplus_i \mathcal{Y}_i = \bigoplus_i \mathcal{Y}_{i1} \otimes \mathcal{Y}_{i2}$ such that for any i :*

1. Π_V and Π_W preserve \mathcal{Y}_i ;
2. Restricted to \mathcal{Y}_i , Π_V and Π_W act non-trivially only on \mathcal{Y}_{i1} and \mathcal{Y}_{i2} respectively.

In other words, V can be decomposed as $V = \bigoplus_i V|_{\mathcal{Y}_{i1}} \otimes \mathcal{Y}_{i2}$, where $V|_{\mathcal{Y}_{i1}} \subseteq \mathcal{X} \otimes \mathcal{Y}_{i1}$, and similarly, W can be decomposed as $W = \bigoplus_i W|_{\mathcal{Y}_{i2}} \otimes \mathcal{Y}_{i1}$, where $W|_{\mathcal{Y}_{i2}} \subseteq \mathcal{Y}_{i2} \otimes \mathcal{Z}$.

Corollary 4.7. *In the setting of Lemma 4.6, if V, W span the whole space, then there exists $W' \subseteq W$ only depending on \mathcal{Y} such that V, W' span the whole space as well.*

Proof. Note that each slice \mathcal{Y}_i is preserved by Π_V and Π_W , so W, V span the space means the restricted W and V on each \mathcal{Y}_i span $\mathcal{X} \otimes \mathcal{Y}_i \otimes \mathcal{Z}$. Furthermore, the restricted V and W share no subqudits, thus either the restricted V or W on \mathcal{Y}_i should be $\mathcal{X} \otimes \mathcal{Y}_i \otimes \mathcal{Z}$. Let S be $\{i : \text{the restricted } W \text{ on } \mathcal{Y}_i \text{ is } \mathcal{X} \otimes \mathcal{Y}_i \otimes \mathcal{Z}\}$ and W' be $\bigoplus_{i \in S} \mathcal{X} \otimes \mathcal{Y}_i \otimes \mathcal{Z}$. Then, it is easy to see that W' satisfies the required conditions. \square

Now, we are ready to present the proof of Lemma 4.5.

Proof (of Lemma 4.5). Given a solitary $j \in [n]$, suppose $\mathcal{N}(j) = [k]$. By applying Lemma 4.6 iteratively, \mathcal{H}_j can be decomposed to several slices, and each slice \mathcal{H}_{jl} consists of k subqudits $\mathcal{H}_{jl1} \otimes \mathcal{H}_{jl2} \otimes \cdots \otimes \mathcal{H}_{jlk}$ such that each restricted V_i where $i \in [k]$ only depends on subqudit \mathcal{H}_{jli} . According to Corollary 4.7, it is not hard to see that each restricted V_i can be replaced with V'_i which acts trivially on all the subqudits in its slice, and the resulted subspace set still span the space. In other words, the qudit \mathcal{H}_j can become classical by proper rotation. Finally, note that $R(V'_i)$ may be strictly smaller than $R(V_i)$, and this problem can be addressed by following the same method with Lemma 4.2.

The following theorem, which says that CLLL is equivalent to VLLL on many of the most common interaction bipartite graphs, is immediate by Lemma 4.5.

Theorem 4.8. *Given an interaction bipartite graph G_B where all the right vertices are solitary, we have $\mathcal{C}\partial(G_B) = \mathcal{V}\partial(G_B)$.*

4.1.2 A Theorem for Gap Decision

We propose a necessary and sufficient condition to decide whether Shearer's bound is tight for CLLL, which will be used in the proofs of theorems about reduction rules. Our condition is a nontrivial commuting analog of the condition for classical VLLL [24].

Theorem 4.9. *Given an interaction bipartite graph G_B and a vector \mathbf{r} of positive reals, the following two conditions are equivalent:*

1. For any rational $\lambda \mathbf{r} \in \mathcal{CI}(G_B) \setminus \mathcal{C}\partial(G_B)$, there is an exclusive subspace set with interaction bipartite graph G_B and relative dimension vector $\lambda \mathbf{r}$.
2. G_B is gapless for CLLL in the direction of \mathbf{r} .

We give the proof of Theorem 4.9 in Appendix A. By this theorem, one can prove gapless just by constructing *exclusive* subspace sets, without computing the critical threshold of CLLL or Shearer's bound. See the proof of Theorem 4.13 as an application of Theorem 4.9.

By Theorem 4.9 we have the following corollary. The proof is given in Appendix A.

Corollary 4.10. *Given an interaction bipartite graph G_B and a rational vector $\mathbf{r} \in \mathcal{C}\partial(G_B)$, if there exists an exclusive subspace set with interaction bipartite graph G_B and relative dimension vector \mathbf{r} , then G_B is gapless in the direction of \mathbf{r} .*

4.1.3 Reduction Rules

To infer gap existence of a bipartite graph from known ones, a set of reduction rules are established for VLLL [24]. With these rules, various bipartite graphs, in particular, combinatorial ones, are shown to be gapful/gapless. In this subsection, we show these reduction rules apply to CLLL as well. Meanwhile, we introduce another operation (the sixth one) which preserves both gapful and gapless. With these operations, the interaction bipartite graph which is a tree can be shown to be gapless immediately.

Given an interaction bipartite graph $G_B = ([m], [n], E_B)$, we consider the following 6 types of operations on G_B :

1. **Delete- R -Leaf:** Delete a vertex $j \in [n]$ on the right side with $|\mathcal{N}(j)| \leq 1$, and remove the incident edge if any.
2. **Duplicate- L -Vertex:** Given a vertex $i \in [m]$ on the left side, add a vertex i' to the left side, and add edges incident to i' so that $\mathcal{N}(i') = \mathcal{N}(i)$.
3. **Duplicate- R -Vertex:** Given a vertex $j \in [n]$ on the right side, add a vertex j' to the right side, and add some edges incident to j' so that $\mathcal{N}(j') \subseteq \mathcal{N}(j)$.
4. **Delete-Edge:** Delete an edge from E_B provided that the base graph G_D remains unchanged.
5. **Delete- L -Vertex:** Delete a vertex $i \in [m]$ on the left side, and remove all the incident edges.
6. **Delete- L -Leaf:** Delete a vertex $i \in [m]$ on the left side with $|\mathcal{N}(i)| \leq 1$, and remove the incident edge if any.

We also define the inverses of the above operations. The inverse of an operation O is the operation O' such that for any G_B , $O'(O(G_B)) = O(O'(G_B)) = G_B$.

The next theorems show how these operations influence the existence of gaps for CLLL.

Theorem 4.11. *A gapless interaction bipartite graph remains gapless for CLLL after applying Delete- L -Vertex or the inverse of Delete-Edge.*

Theorem 4.12. *A gapful interaction bipartite graph remains gapful for CLLL after applying Delete-Edge or the inverse of Delete- L -Vertex.*

The above two theorems are trivial, and the proofs are omitted.

Theorem 4.13. *An interaction bipartite graph $G_B = ([m], [n], E_B)$ is gapful for CLLL, if and only if it is gapful after applying Delete- L -Leaf, Delete- R -Leaf, Duplicate- L -Vertex, Duplicate- R -Vertex, or their inverse operations.*

See Appendix B for the proof of Theorem 4.13. With a similar argument, we can also prove that Delete- L -Leaf maintains gapless/gapful for VLLL.

Theorem 4.14. *An interaction bipartite graph G_B is gapful for VLLL, if and only if it remains gapful after applying Delete- L -Leaf or its inverse operation.*

With these reduction rules, it is easy to see all trees are gapless, which include 1-D chains [36], regular trees [9, 25, 39, 44], the treelike bipartite graphs [24].

Theorem 4.15. *An interaction bipartite graph G_B is gapless for CLLL if G_B is a tree.*

Proof. Applying Delete- L -Leaf or Delete- R -Leaf on G_B repeatedly results in an interaction bipartite graph $G'_B = ([1], [1], \{(1, 1)\})$. Obviously, G'_B is gapless, which implies G_B is gapless as well by Theorem 4.13. \square

4.2 Tight Region for Trees

In the above section, we have proved that trees are gapless. In this section, we calculate the tight region of LLLs on trees explicitly. For CLLL, our results also apply to the case where the dimensions of qudits are specified.

The interaction bipartite graph which is a tree includes two interesting families of bipartite graphs, the treelike bipartite graphs defined in [24] and the regular trees investigated in [9, 25, 39]. He et al. [24] have already calculated the tight regions of treelike bipartite graphs for VLLL. Here we extend the classical result to the commuting case on a larger family of graphs even if the dimensions of qudits are given.

Given an interaction bipartite graph $G_B = ([m], [n], E_B)$ which is a tree and the dimensions of qudits, without loss of generality, we can assume that the root is the right vertex. Furthermore, we can also assume that the leaves of the tree are right vertices as well, because adding right vertices as leaves and setting the dimensions of corresponding qudits as one do not change the boundary.

Theorem 4.16. *Given an interaction bipartite graph $G_B = ([m], [n], E_B)$ which is a tree, a rational vector \mathbf{r} and an integer vector \mathbf{d} , for any vertex k in G_B , let \mathcal{C}_k be the set of children of k . Then, all commuting subspace sets of the setting $(G_B, \leq \mathbf{r}, \mathbf{d})$ are frustration free if and only if there exists $\mathbf{q} = (q_1, \dots, q_n)$ where $q_j < d_j$ and*

$$q_j = \begin{cases} 0 & \text{if vertex } j \text{ is a leaf of } G_B, \\ \sum_{i \in \mathcal{C}_j} \lfloor r_i \cdot d_j \cdot \prod_{j' \in \mathcal{C}_i} \frac{d_{j'}}{d_{j'} - q_{j'}} \rfloor & \text{otherwise.} \end{cases} \quad (2)$$

Proof. The proof is by induction on m . Basic: The case for $m = 1$ holds trivially.

Induction: Let j^* be a right vertex in the tree whose descendants in $[n]$ are leaves.

\implies : Suppose such \mathbf{q} does not exist on G_B . Let T_j be the subtree rooted at j for $j \in [n]$. Note that $q_j = 0 < d_j$ for any leaf j of G_B , then there must be some $j' \in [n]$ which is an ancestor of j^* or j^* itself, such that the \mathbf{q} can be defined on any subtree of $T_{j'}$ except $T_{j'}$ itself. According to whether $j' = j^*$ or not, it is divided into cases.

Case 1: If $j' = j^*$, we have $q_{j^*} = \sum_{i \in \mathcal{C}_{j^*}} \lfloor r_i d_{j^*} \prod_{j \in \mathcal{C}_i} \frac{d_j}{d_j - q_j} \rfloor = \sum_{i \in \mathcal{C}_{j^*}} \lfloor r_i d_{j^*} \rfloor \geq d_{j^*}$, then there exists commuting $\{\mathcal{H}_{j^*}^i\}_{i \in \mathcal{C}_{j^*}}$ where $\mathcal{H}_{j^*}^i \subseteq \mathcal{H}_{j^*}$, $\dim(\mathcal{H}_{j^*}^i) = \lfloor r_i d_{j^*} \rfloor$ such that $\bigoplus_{i \in \mathcal{C}_{j^*}} \mathcal{H}_{j^*}^i = \mathcal{H}_{j^*}$. For each $i \in \mathcal{C}_{j^*}$, we define V_i as $\mathcal{H}_{j^*}^i \otimes \mathcal{H}_{[n] \setminus j^*}$. Thus, we have $R(V_i) = \lfloor r_i d_{j^*} \rfloor / d_{j^*} \leq r_i$ and $\{V_i\}_{i \in \mathcal{C}_{j^*}}$ spans $\mathcal{H}_{[n]}$.

Case 2: If $j' \neq j^*$, we have $q_{j^*} < d_{j^*}$. Decompose \mathcal{H}_{j^*} to two orthogonal subspaces $\mathcal{H}_{j^*}^a \oplus \mathcal{H}_{j^*}^b$ where $\dim(\mathcal{H}_{j^*}^a) = \sum_{i \in \mathcal{C}_{j^*}} \lfloor r_i d_{j^*} \rfloor$ and $\dim(\mathcal{H}_{j^*}^b) = d_{j^*} - \dim(\mathcal{H}_{j^*}^a)$. Note that $\dim(\mathcal{H}_{j^*}^b) > 0$, so there is such a decomposition of \mathcal{H}_{j^*} . Let $S \subset [n]$ be the right vertices in $T' := (T_{j'} \setminus T_{j^*}) \cup \{j^*\}$. Define \mathbf{d}' as the dimension vector of \mathcal{H}'_S where $\mathcal{H}'_{j^*} = \mathcal{H}_{j^*}^b$ and \mathcal{H}'_j is \mathcal{H}_j for any $j \in S \setminus \{j^*\}$. In addition, let \mathbf{r}' be the induced \mathbf{r} on T' except by letting $r'_{i^*} = r_{i^*} \cdot \frac{d_{j^*}}{d'_{j^*}}$, where i^* is the father of j^* , so the dimensions of these subspaces keep the same. Let \mathbf{q}' be defined as (2) according to $T', \mathbf{r}', \mathbf{d}'$. Thus, $q'_{j^*} = 0$. We

have $d'_{j^*} - q'_{j^*} = d_{j^*} - \sum_{i \in \mathcal{C}_{j^*}} \lfloor r_i d_{j^*} \rfloor = d_{j^*} - q_{j^*}$. Then, it is easy to check that $q'_j = q_j$ for any $j \in S \setminus \{j^*\}$. By the induction hypothesis, the setting $(T', \leq \mathbf{r}', \mathbf{d}')$ admits a commuting subspace set \mathcal{V}' spanning $\mathcal{H}_{j^*}^b \otimes \mathcal{H}_{[n] \setminus j^*}$. Moreover, similar to Case 1, we can construct commuting $\{V_i\}_{i \in \mathcal{C}_{j^*}}$ with $R(V_i) \leq r_i$ spanning $\mathcal{H}_{j^*}^a \otimes \mathcal{H}_{[n] \setminus j^*}$. Therefore, $\mathcal{V} = \mathcal{V}' \cup \{V_i\}_{i \in \mathcal{C}_{j^*}}$ spans $\mathcal{H}_{[n]}$.

\Leftarrow : Suppose the setting $(G_B, \mathbf{r}', \mathbf{d})$ for some $\mathbf{r}' \leq \mathbf{r}$ admits a commuting subspace set \mathcal{V} spanning $\mathcal{H}_{[n]}$. If $d_{j^*} \leq \sum_{i \in \mathcal{C}_{j^*}} \lfloor r_i d_{j^*} \rfloor$, then q_{j^*} defined as (2) under the parameter $G_B, \mathbf{r}, \mathbf{d}$ would violate the constraint that $q_{j^*} < d_{j^*}$, which concludes the proof. In the following, we assume $d_{j^*} > \sum_{i \in \mathcal{C}_{j^*}} \lfloor r_i d_{j^*} \rfloor$. Since all right vertices are solitary, it is lossless to assume that all the subspaces in \mathcal{V} can be diagonal w.r.t. the computational basis. Let \mathcal{V}' be the induced \mathcal{V} on $T' := (G_B \setminus T_{j^*}) \cup \{j^*\}$. We claim that \mathcal{V}' spans $\mathcal{H}_{j^*}^c \otimes \mathcal{H}_{[n] \setminus j^*}$ where $\mathcal{H}_{j^*}^c$ is a subspace of \mathcal{H}_{j^*} with $\dim(\mathcal{H}_{j^*}^c) \geq d_{j^*} - \sum_{i \in \mathcal{C}_{j^*}} \lfloor r_i d_{j^*} \rfloor$. Let $\mathcal{H}_{j^*}^d \subseteq \mathcal{H}_{j^*}^c$ be diagonal w.r.t. the computational basis and of dimension $d_{j^*} - \sum_{i \in \mathcal{C}_{j^*}} \lfloor r_i d_{j^*} \rfloor$ and let \mathcal{V}'' be the vector obtained from \mathcal{V}' by letting $V_j'' = V_j' \cap (\mathcal{H}_{j^*}^d \otimes \mathcal{H}_{[n] \setminus j^*})$. So \mathcal{V}'' is a non-frustration free commuting instance of the setting $(T', \leq \mathbf{r}'', \mathbf{d}')$, where \mathbf{d}' is the induced \mathbf{d} on T' except by letting $d'_{j^*} = d_{j^*} - \sum_{i \in \mathcal{C}_{j^*}} \lfloor r_i d_{j^*} \rfloor$, and \mathbf{r}'' is the induced \mathbf{r} on T' except by letting $r''_{i^*} = r_{i^*} \cdot \frac{d_{j^*}}{d'_{j^*}}$ where i^* is the father of j^* to keep the dimensions of these subspaces unchanged. By the induction hypothesis, \mathbf{q} cannot be defined according to (2) under the parameter $T', \mathbf{r}'', \mathbf{d}'$, it is easy to see that so neither does it under the parameters $G_B, \mathbf{r}, \mathbf{d}$, which concludes the proof.

To see why the claim holds, for $i \in \mathcal{C}_{j^*}$, let $\mathcal{H}_{j^*}(i) = \text{span}(\{|e\rangle \in \mathcal{H}_{j^*} : |e\rangle \otimes \mathcal{H}_{[n] \setminus j^*} \subseteq V_i\})$, it is easy to see that $\dim(\mathcal{H}_{j^*}(i)) \leq \lfloor r_i \cdot d_{j^*} \rfloor$. Further, let $\mathcal{H}_{j^*}^c$ be the orthogonal complementary subspace of $(\bigoplus_{i \in \mathcal{C}_{j^*}} \mathcal{H}_{j^*}(i))$ in \mathcal{H}_{j^*} , and we have $\mathcal{H}_{j^*}^c \otimes \mathcal{H}_{[n] \setminus j^*}$ should be spanned by \mathcal{V}' . \square

It is easy to see that $\mathcal{QI}(G_B) \subseteq \mathcal{CI}(G_B) \subseteq \mathcal{VI}(G_B)$ for any G_B . Then by the result on VLLL in [24], we can obtain the tight regions for VLLL, CLLL and QLLL on trees, ignoring the dimensions of qudits.

Theorem 1.4. *For any interaction bipartite graph $G_B = ([m], [n], E_B)$ which is a tree, we have $\mathcal{VI}(G_B) = \mathcal{CI}(G_B) = \mathcal{QI}(G_B) = \mathcal{I}(G_D(G_B))$. Given $\mathbf{r} \in (0, 1)^m$, $\mathbf{r} \in \mathcal{VI}(G_B)$ if and only if there exists $\mathbf{q} \in [0, 1]^n$ where $q_j = 0$ if j is a leaf of G_B and $q_j = \sum_{i \in \mathcal{C}_j} r_i \cdot \prod_{k \in \mathcal{C}_i} \frac{1}{1 - q_k}$ for other $j \in [n]$.*

4.3 Beyond Shearer's Bound for Graphs Containing Cyclic Graphs

In this section, we show that many interaction bipartite graphs are gapful for CLLL. An easy observation is that an interaction bipartite graph G_B is gapless for CLLL if it is gapless for VLLL. Thus, the combinatorial interaction bipartite graph gapless for VLLL defined in [24] are also gapless for CLLL.

Definition 4.6 (Combinatorial interaction bipartite graph [24]). Given two positive integers $m < n$, let $G_{n,m} = ([\binom{n}{m}], [n], E_{n,m})$ where $(i, j) \in E_{n,m}$ if and only if j is in the m -sized subset of $[n]$ represented by i . $G_{n,m}$ is called the (n, m) -combinatorial interaction bipartite graph.

Corollary 4.17. *For $n \geq 4$, $G_{n,n-1}$ is gapless.*

Corollary 4.18. *For any constant m , when n is large enough, $G_{n,n-m}$ is gapless.*

On the other side, it can be proved that all cyclic graphs are gapful. As it is lossless to assume that any pair of left vertices sharing at most one right vertex for cyclic interaction bipartite graphs, by Theorem

4.8 we have the tight regions of CLLL and VLLL are the same for cyclic graphs. Thus, we can obtain the tight region of CLLL for cyclic interaction bipartite graph from the tight region of VLLL [24]. Meanwhile, because it has been proved that cyclic graphs are gapful for VLLL, we immediately have the following corollary.

Corollary 4.19. *Cyclic interaction bipartite graphs are gapful.*

By the definition of *contained graph* (see Definition 2.10), it is easy to verify that if an interaction bipartite graph G_B containing another G'_B , then we can obtain G'_B from G_B by applying Delete- L -Vertex and Delete- R -Leaf iteratively. Thus, by Theorem 4.13 and Theorem 4.11, an interaction bipartite graph G_B is gapful if it contains a gapful one. According to Corollary 4.19, we obtain the following result.

Theorem 1.6 (Restated). *Any interaction bipartite graphs containing a cyclic one is gapful.*

It is easy to verify that any interaction bipartite graph contains a cyclic one if its base graph has an induced cycle of length at least four. Thus, by Theorems 4.15 and 1.6, we have the following corollary, which almost gives a complete characterization of gapful/gapless for CLLL except when the base graph has only 3-cliques.

Corollary 1.7. *Given an interaction bipartite graph, Shearer's bound is tight for CLLL if its base graph is a tree, and is not tight if its base graph is not a chordal graph.*

5 Critical Thresholds for LLLs

We study the critical thresholds for the four kinds of LLLs in this section. The main result is Theorem 1.8, which provides lower bounds for the gaps between the critical thresholds of LLLs. Here, we illustrate our main idea with the proof of lower bound on $P_V - P_A$. Our argument consists of three steps.

Given an interaction bipartite graph G_B and an event set which is unavoidable, we first show that there exist a pair of adjacent events with considerable intersections on each 2-discrete cyclic graph, namely Theorem 5.10. Second, for such pairs, we can “cut” down the intersections and reduce the probabilities of events while keeping the resulted probability vector beyond Shearer's bound. However, the resulted probability vector is possibly not symmetric. Then, we develop a theorem to depict how probabilities can transfer between events, with which we can make the probability vector symmetric again.

The main difficulty is how to show the reduced probability vector is still beyond Shearer's bound. We choose the pairs of events for cutting carefully (i.e., we choose the compatible pairs). Then, we employ the structure properties developed in [24] and [17] to show that $G_D(G_B)$ is still the lopsidedependency graph of the resulted events. Thus, with lopsidedependency LLL, we deduce that the resulted probability vector is still beyond Shearer's bound.

The above argument can be extended to lower bound $R_C - P_A$ naturally by using Lemma 4.5, which says we can treat solitary qudits as classic variables.

5.1 Tools

In this subsection, we introduce three useful tools, namely the lopsidedependency condition [12], the discreteness theorem [17, 24] and the monotonicity structure [17]. With these tools, we introduce the cutting operation, which plays a central role in our proof.

Lopsidedependency Condition The “lopsidedependency” condition, which is generalized from dependency by Erdős and Spencer [12], has enabled several interesting applications of LLL in combinatorics and theoretical computer science [16, 31].

Definition 5.1 (Lopsidedependency). A lopsidedependency graph is an undirected graph $G_D = ([m], E_D)$ such that for any vertex $i \in [m]$ and any $K \subseteq [m] \setminus \Gamma_i^+$, $\mathbb{P}(A_i | \bigcup_{k \in K} A_k) \geq \mathbb{P}(A_i)$ holds.

Shearer’s condition also holds for the lopsidedependency graph.

Theorem 5.1 ([44]). For a lopsidedependency graph $G_D = (V, E)$ and probabilities $\mathbf{p} \in \mathbb{R}^{|V|}$ the following conditions are equivalent:

1. \mathbf{p} is in Shearer’s bound for G_D .
2. for any probability space Ω and events $\{A_v \subseteq \Omega : v \in V\}$ having G_D as lopsidedependency graph and satisfying $\mathbb{P}(A_v) \leq p_v$, we have $\mathbb{P}(\bigcup_{v \in V} A_v) \geq I(G_D, \mathbf{p}) > 0$.

Discreteness Theorem The discreteness theorem, established by Kun He et al. [24] and András Gilyén [17] independently, is a very useful structural result on VLLL. Given an event-variable graph $G_B = ([m], [n], E_B)$, recall that each event A_i where $i \in [m]$ is fully determined by some subset \mathcal{X}_i of a set of mutually independent random variables $\mathcal{X} = \{X_1, \dots, X_n\}$. We use Cartesian product $\prod_{j=1}^n \{x_j\}$ to denote the sample $X_1 = x_1, \dots, X_n = x_n$ in the sample space. Similarly, we will also use $U_{X_1} \times (\prod_{j=2}^n \{x_j\})$ to denote the samples $X_2 = x_2, \dots, X_n = x_n$, recalling that U_{X_i} is the universal set of the possible values of X_i .

Definition 5.2 (d-discrete event set). Given a bipartite graph $G_B = ([m], [n], E)$, an event set $\mathcal{A} \sim G_B$ is called **d-discrete event set** where $\mathbf{d} = (d_1, \dots, d_n)$ if the following conditions hold:

1. for any $j \in [n]$, there are d_j different values $x_j^1, \dots, x_j^{d_j}$ such that $\sum_{i=1}^{d_j} \mathbb{P}(X_j = x_j^i) = 1$.
2. for any $i \in [m]$, A_i is $\bigcup_{(k_1, \dots, k_n) \in S} \prod_{j=1}^n \{x_j^{k_j}\}$ where S is some subset of $[d_1] \times \dots \times [d_n]$.

Theorem 5.2 (Discreteness Theorem [24]). Given a bipartite graph $G_B = ([m], [n], E)$ and $\mathbf{p} \in \mathcal{V}\partial(G_B)$, let $\mathbf{d} = (d_1, \dots, d_n)$ where d_j is the degree of the vertex $j \in [n]$ in G_B . Then there is a **d-discrete event set** $\mathcal{A} \sim G_B$ such that $\mathbb{P}(\mathcal{A}) = \mathbf{p}$ and $\mathbb{P}(\bigcup_{A \in \mathcal{A}} A) = 1$.

The following is a generalization of the discreteness theorem to CLLL. The idea is to replace probability with relative dimension, and the main steps appear similar, so we omit the proof.

Theorem 5.3 (Discreteness Theorem for CLLL). Given a bipartite graph $G_B = ([m], [n], E)$ and $\mathbf{r} \in \mathcal{C}\partial(G_B)$, let $\mathbf{d} = (d_1, \dots, d_n)$ where d_j is the degree of the vertex $j \in [n]$ in G_B . Then there is a commuting subspace set $\mathcal{V} \sim G_B$ of relative dimensions \mathbf{r} spanning the whole space and satisfying that for any solitary $j \in [n]$, \mathcal{H}_j is classical and takes d_j different values.

Monotonicity Structure Suppose the degree of variable X is 2. Then according to discreteness theorem, it is lossless to assume $U_X = \{x^1, x^2\}$. Besides, András Gilyén [17] shows another nice structure about X for VLLL.

Definition 5.3 (Cross section event and X -monotone for VLLL). Given a event A and a variable X , define the *cross section event* A^{x^k} as the event satisfying $\{x^k\} \times A^{x^k} = A \cap (\{x^k\} \times U)$. Informally, A^{x^k} is the cross section of A on plane $X = x^k$. For $U_X = \{x^1, x^2\}$, we say A is X -up (or X -down resp.) if $A^{x^1} \subseteq A^{x^2}$ (or $A^{x^2} \subseteq A^{x^1}$ resp.) holds. We say A is X -monotone if it is either X -up or X -down. Note that if A is independent of X , A is both X -up and X -down, and thus X -monotone.

Definition 5.4 (Standard event set). Given a bipartite graph $G_B = ([m], [n], E)$, let $\mathbf{d} = (d_1, \dots, d_n)$ where d_j is the degree of the vertex $j \in [n]$ in G_B . An event set \mathcal{A} is called a standard event set of G_B if the following conditions hold:

1. \mathcal{A} is \mathbf{d} -discrete and $\mathcal{A} \sim G_B$.
2. For any $j \in [n]$ where $d_j = 2$, the two associated events are both X_j -monotone but in the opposite directions.

Theorem 5.4 ([17]). *Given a bipartite graph $G_B = ([m], [n], E)$ and $\mathbf{p} \in \partial_v(G_B)$, there is a standard event set \mathcal{A} of G_B such that $\mathbb{P}(\mathcal{A}) = \mathbf{p}$ and $\mathbb{P}(\bigcup_{A \in \mathcal{A}} A) = 1$.*

The intuition of theorem 5.4 is as follows. For any variable X_j with degree 2, let A_{i_1} and A_{i_2} be the associated events of X_j . To maximize $\mathbb{P}(\bigcup_{A \in \mathcal{A}} A)$, A_{i_1} and A_{i_2} tend to be exclusive, i.e., X -monotone in the opposite directions.

This structural result can be generalized to the commuting case naturally. The proof is similar and omitted here.

Definition 5.5 (Cross section subspace and X -monotone for CLLL). Given a subspace V and a qudit X which is classical, define the *cross section subspace* V^{x^k} as the subspace satisfying $|x^k\rangle \otimes V^{x^k} = V \cap (|x^k\rangle \otimes \mathcal{H})$, where \mathcal{H} stands for the Hilbert space spanned by the other qudits. Similarly, we also define X -up and X -down.

Theorem 5.5. *Given a bipartite graph $G_B = ([m], [n], E)$ and $\mathbf{r} \in \mathcal{C}\partial(G_B)$, there is a standard commuting subspace set $\mathcal{V} \sim G_B$ such that $R(\mathcal{V}) = \mathbf{r}$ and $R(\sum_{i \in [m]} V_i) = 1$. Here, we say \mathcal{V} is a standard subspace set of G_B if*

1. For any solitary $j \in [m]$, \mathcal{H}_j is classical and takes d_j different values.
2. For any classical \mathcal{H}_j where $d_j = 2$, the two associated subspaces are both X_j -monotone but in the opposite directions.

Besides Theorems 5.4 and 5.5, we can also prove the following properties about X -monotone, which will be used in the proof of Lemma 5.8.

Lemma 5.6. *Let X, X_1, \dots, X_k be variables each taking 2 different values, and A_1 and A_2 are two events.*

- (a) *If A_1 and A_2 are both X -monotone in the same direction, then $A_1 \cup A_2$ and $A_1 \cap A_2$ are also X -monotone in the same direction.*
- (b) *If A_1 and A_2 are both X -monotone in the opposite directions, then $A_1 \setminus A_2$ is X -monotone in the opposite directions with A_2 .*

(c) Suppose X_1, \dots, X_k are all the variables shared by A_1 and A_2 , if for each $i \in [k]$, A_1 and A_2 are both X_i -monotone in the same direction, then $\mathbb{P}(A_1|A_2) \geq \mathbb{P}(A_1)$.

Proof. Part (a) and (b) are obvious, and we omit the proof.

Part (c). The proof is by induction on k . Basic: when $k = 1$, we have

$$\begin{aligned}
\mathbb{P}(A_1 \cap A_2) &= \sum_{i \in [2]} \mathbb{P}(A_1 \cap A_2 \cap \{X_1 = x_1^i\}) = \sum_{i \in [2]} \mathbb{P}[(A_1 \cap A_2)^{x_1^i}] \mathbb{P}(X_1 = x_1^i) \\
&= \sum_{i \in [2]} \mathbb{P}(A_1^{x_1^i} \cap A_2^{x_1^i}) \mathbb{P}(X_1 = x_1^i) = \sum_{i \in [2]} \mathbb{P}(A_1^{x_1^i}) \mathbb{P}(A_2^{x_1^i}) \mathbb{P}(X_1 = x_1^i) \\
&= \sum_{i \in [2]} (\mathbb{P}(A_1^{x_1^i}) \mathbb{P}(A_2^{x_1^i}) \mathbb{P}(X_1 = x_1^i) \sum_{j \in [2]} \mathbb{P}(X_1 = x_1^j)) \\
&= \sum_{i \in [2]} (\mathbb{P}(A_1^{x_1^i}) \mathbb{P}(X_1 = x_1^i) (\sum_{j \in [2]} \mathbb{P}(A_2^{x_1^j}) \mathbb{P}(X_1 = x_1^j))) \\
&\geq (\sum_{i \in [2]} \mathbb{P}(A_1^{x_1^i}) \mathbb{P}(X_1 = x_1^i)) \times (\sum_{j \in [2]} \mathbb{P}(A_2^{x_1^j}) \mathbb{P}(X_1 = x_1^j)) = \mathbb{P}(A_1) \mathbb{P}(A_2).
\end{aligned} \tag{3}$$

The inequality is due to that A_1 and A_2 are X_1 -monotone in the same direction.

Induction: Suppose that the case for $k \leq t$ has already been shown. Now we prove it also holds for $k = t + 1$. For each $i \in [2]$, note that $A_1^{x_1^i}$ and $A_2^{x_1^i}$ are both monotone with X_2, X_3, \dots, X_{t+1} in the same direction. Thus, by the induction hypothesis, we have

$$\begin{aligned}
\mathbb{P}(A_1 \cap A_2) &= \sum_{i \in [2]} \mathbb{P}(A_1^{x_1^i} \cap A_2^{x_1^i}) \mathbb{P}(X_1 = x_1^i) \\
&\geq \sum_{i \in [2]} \mathbb{P}(A_1^{x_1^i}) \mathbb{P}(A_2^{x_1^i}) \mathbb{P}(X_1 = x_1^i) \\
&\geq \mathbb{P}(A_1) \mathbb{P}(A_2),
\end{aligned}$$

where the second inequality has been shown in (3). □

By Theorem 5.6, we have the following corollary.

Corollary 5.7. Let $\{A_i\}$ and $\{B_k\}$ be two disjoint sets of events and $(\bigcup_i \text{vbl}(A_i)) \cap (\bigcup_k \text{vbl}(B_k)) = \{X_1, \dots, X_t\}$. If for each $j \in [t]$, only one event in $\{A_i\}$ and one event in $\{B_k\}$ depend on X_j , and are both X_j -monotone in the same direction, then $\bigcup_i A_i$ and $\bigcup_k B_k$ are positively correlated, i.e., $\mathbb{P}(\bigcup_i A_i | \bigcup_k B_k) \geq \mathbb{P}(\bigcup_i A_i)$.

Cutting Operation Armed with the above tools, we are ready to introduce one key ingredient of our approach – *the cutting operation*: Given a bipartite graph $G_B = ([m], [n], E)$ and a standard event set $\mathcal{A} \sim G_B$, we say an event A_i or left vertex i is *2-discrete* if the degree of each variable associated with A_i is at most 2, that is, $|\mathcal{N}(j)| \leq 2$ holds for any $j \in \mathcal{N}(i)$. Additionally, for any event A_k adjacent with A_i where A_i is *2-discrete*, the so-called (k, i) -cutting is simply cutting A_k down to $A'_k = A_k \setminus A_i$. We say (k_1, i_1) -cutting and (k_2, i_2) -cutting are compatible if $\mathcal{N}(i_1) \cap \mathcal{N}(i_2) = \emptyset$, and a set of cutting operations \mathcal{S} is compatible if any pair of cuttings are compatible.

For any compatible set of cuttings \mathcal{S} , applying \mathcal{S} in parallel on standard \mathcal{A} results in a new event set \mathcal{A}' where $A'_k = A_k \setminus (\bigcup_{i: (k,i) \in \mathcal{S}} A_i)$ for any $A'_k \in \mathcal{A}'$.

Without loss of generality, let $\mathcal{X} = \{X_1, X_2, \dots, X_n\}$ be the set of variables. The follows are some useful properties of the cutting operation.

Lemma 5.8. Let $\mathcal{S} = \{(k_l, i_l)\}$ be a compatible set of cutting operations. Applying \mathcal{S} in parallel on standard \mathcal{A} leads to a new event set \mathcal{A}' . Then, the following holds:

(a) if $(k, i) \in \mathcal{S}$, then $(i, t) \notin \mathcal{S}$ for any t and $(k', i) \notin \mathcal{S}$ for any $k' \neq k$.

(b) for any l , $A'_{i_l} = A_{i_l}$;

(c) Let $\mathcal{N}(j)$ be $\{i_l, t\}$ for some l where $t \neq k_l$, then there are exact three events A'_{i_l} , A'_{k_l} and A'_t depending on X_j . Moreover, A'_{i_l} is X_j -monotone, say X_j -up, and A'_{k_l} and A'_t are both X_j -down.

Proof. Part (a). If $(i, t) \in \mathcal{S}$, by the definition of cutting operation, A_t and A_i are adjacent. Hence $\mathcal{N}(i) \cap \mathcal{N}(t) \neq \emptyset$, a contradiction to the definition of compatibility. Similarly, we can also prove that $(k', i) \notin \mathcal{S}$.

Part (b). According to Part (a), $(i_l, t) \notin \mathcal{S}$ for any t , thus A_{i_l} would not be cut. Hence, we have for any l , $A'_{i_l} = A_{i_l}$.

Part (c). By the definition of compatibility, $j \notin \mathcal{N}(i_{l'})$ for any $l' \neq l$. Combined this with $\text{vbl}(A'_k) = \text{vbl}(A_k) \cup (\bigcup_{i:(k,i) \in \mathcal{S}} \text{vbl}(A_i))$, we have there are exact three events A'_{i_l} , A'_{k_l} and A'_t depending on X_j . By Part (b), we have $A'_{i_l} = A_{i_l}$, and is X_j -monotone. W.l.o.g., let's assume that A_{i_l} is X_j -up. Then, A_{k_l} , A_t are both X_j -down. Note that $A'_{k_l} = (A_{k_l} \setminus A_{i_l}) \setminus (\bigcup_{i \neq i_l: (k_l, i) \in \mathcal{S}} A_i)$ and $A'_t = A_t \setminus (\bigcup_{i: (t, i) \in \mathcal{S}} A_i)$, thus they are still X_j -down according to Lemma 5.6 (b). \square

The point of cutting operation is to reduce the probabilities of events while keeping the resulted probabilities still beyond Shearer's bound. This is shown in the following theorem.

Theorem 5.9. Given $G_B = ([m], [n], E_B)$ and a standard event set \mathcal{A} , applying a compatible set of cuttings $\mathcal{S} = \{(k_l, i_l)\}$ in parallel leads to a new event set \mathcal{A}' satisfying

(a) $\bigcup_{i \in [m]} A'_i = \bigcup_{i \in [m]} A_i$, and

(b) For any disjoint sets $K_1, K_2 \subseteq [m]$ where $(\bigcup_{i \in K_1} \mathcal{N}(i)) \cap (\bigcup_{i \in K_2} \mathcal{N}(i)) = \emptyset$, we have $\bigcup_{k \in K_1} A'_k$ and $\bigcup_{k \in K_2} A'_k$ are positively correlated. Thus G_D is a lopsidedependency graph of \mathcal{A}' .

Proof. Part (a). Since $A'_{i_l} = A_{i_l}$ for any l , we have

$$\begin{aligned} A'_{k_l} \cup \left(\bigcup_{i: (k_l, i) \in \mathcal{S}} A'_i \right) &= \left(A_{k_l} \setminus \left(\bigcup_{i: (k_l, i) \in \mathcal{S}} A_i \right) \right) \cup \left(\bigcup_{i: (k_l, i) \in \mathcal{S}} A'_i \right) \\ &= \left(A_{k_l} \setminus \left(\bigcup_{i: (k_l, i) \in \mathcal{S}} A_i \right) \right) \cup \left(\bigcup_{i: (k_l, i) \in \mathcal{S}} A_i \right) \\ &= A_{k_l} \cup \left(\bigcup_{i: (k_l, i) \in \mathcal{S}} A_i \right), \end{aligned}$$

which implies $\bigcup_{i \in [m]} A_i = \bigcup_{i \in [m]} A'_i$ by noting that only the events in $\{A_{k_l}\}$ would be cut.

Part (b). Observe that $\text{vbl}(A'_k) = \text{vbl}(A_k) \cup (\bigcup_{i: (k, i) \in \mathcal{S}} \text{vbl}(A_i))$. Thus, given any variable X_j shared by $\bigcup_{k \in K_1} A'_k$ and $\bigcup_{k \in K_2} A'_k$, we have $\mathcal{N}(j) = \{i_l, t\}$ for some $l, k_l \in K_1, t \in K_2$, and $i_l \notin K_1 \cup K_2$, otherwise, $(\bigcup_{i \in K_1} \mathcal{N}(i)) \cap (\bigcup_{i \in K_2} \mathcal{N}(i)) \neq \emptyset$. (Or $k_l \in K_2, t \in K_1$ and $i_l \notin K_1 \cup K_2$.) According to Lemma 5.8 (c), there are exact three events A'_{i_l} , A'_{k_l} , A'_t depending on X_j , and A'_t and A'_{k_l} are both X_j -monotone in the same direction. Therefore, $\bigcup_{k \in K_1} A'_k$ and $\bigcup_{k \in K_2} A'_k$ are positively correlated by Corollary 5.7. \square

5.2 Cycle and Intersection

To reduce the probabilities of events with the cutting operation, we need to find some neighboring events with a considerable intersection. Indeed, the cyclic structure would guarantee the existence of such events.

As we only care about the tight regions of LLLs, it is lossless to assume l -cyclic bipartite graph is the canonical one $C_l = ([l], [l], E_l)$ where $E_l = \{(i, i), (i, i+1) : i \in [l-1]\} \cup \{(l, l), (l, 1)\}$.

Definition 5.6 (2-discrete Cyclic Graphs). Given a bipartite graph $G_B = ([m], [n], E)$, let C be any cyclic graph contained in G_B . We call C is 2-discrete if any right vertex j on C is of degree at most 2. Let $\mathcal{C}(G_B)$ be the sets of 2-discrete cyclic graphs contained in G_B .

The following theorem shows that there are some neighboring events with a considerable intersection on each $C \in \mathcal{C}(G_B)$.

Theorem 5.10. *Given a bipartite graph $G_B = ([m], [n], E_B)$, let $p = P_V(G_B)$ and $\mathbf{p} = (p, p, \dots, p)$. Then there is a standard event set \mathcal{A} of G_B such that $\mathbb{P}(\mathcal{A}) = \mathbf{p}$, $\mathbb{P}(\bigcup_{A \in \mathcal{A}} A) = 1$ and for any 2-discrete cyclic graph $C \in \mathcal{C}(G_B)$, there are three events A_i, A_j, A_k on C where A_j is neighboring to A_i and A_k and $\mathbb{P}(A_j \cap (A_i \cup A_k)) \geq p^3$.*

Proof. The proof is by induction on $|\mathcal{C}(G_B)|$. Basic: The case for $|\mathcal{C}(G_B)| = 0$ is exactly Theorem 5.4.

Induction: Suppose that the theorem has been proved for any G'_B where $|\mathcal{C}(G'_B)| \leq N$. Now we consider the bipartite graph G_B where $|\mathcal{C}(G_B)| = N + 1$. Let C_l be any 2-discrete cyclic graph in $\mathcal{C}(G_B)$, and \mathcal{A} be a standard event set. W.l.o.g., let A_1, A_2, \dots, A_l be the events and X_1, X_2, \dots, X_l be the variables on C_l . From the definition of ‘‘containing’’, we have all the other variables are not shared by any A_i, A_j where $i, j \in [l]$. Thus, we can assume that for any $i \in [l]$, $\text{vbl}(A_i) = \{X_i, X_{i+1}, Y_i\}$ where $Y_i \in [0, 1]$ stands for all the other variables of A_i except X_i, X_{i+1} without loss of generality.

Since C_l is 2-discrete, we have X_i is only shared by two events in G_B for any $i \in [l]$. Then by the discreteness of \mathcal{A} , we can assume that each X_i takes value from $U = \{0, 1\}$. Because A_i is both X_i - and X_j -monotone in certain directions, it is easy to see $A_i^{y_i}$ can only be one of the following six events. And according to which event $A_i^{y_i}$ is, we divide $U_{Y_i} = [0, 1]$ into five sets. Here, a, b are constants in $\{0, 1\}$.

- (i) $A_i^{y_i} = \emptyset$ or $\{a\} \times \{b\}$, then $y_i \in I_i^0$;
- (ii) $A_i^{y_i} = \{a\} \times U$, then $y_i \in I_i^1$;
- (iii) $A_i^{y_i} = U \times \{b\}$, then $y_i \in I_i^2$;
- (iv) $A_i^{y_i} = (\{a\} \times U) \cup (U \times \{b\})$, then $y_i \in I_i^3$;
- (v) $A_i^{y_i} = U \times U$, then $y_i \in I_i^4$.

Without loss of generality, let I_i^0 be the interval $[0, \mathbb{P}(Y_i \in I_i^0)]$ for any $i \in [l]$. To simplify notation, let $I_i^{>0}$ denote $I_i^1 \cup I_i^2 \cup I_i^3 \cup I_i^4$. For any $y_i^1 \in I_i^0$ and $y_i^2 \in I_i^{>0}$, it is easy to see $A_i^{y_i^1} \subseteq \{a\} \times \{b\} \subseteq A_i^{y_i^2}$. In addition, note that $\int_{y_i \in I_i^{>0}} \mathbb{P}(A_i^{y_i}) + \int_{y_i \in I_i^0} \mathbb{P}(A_i^{y_i}) = \mathbb{P}(A_i) = p$, then we have

$$\mathbb{P}(A_i \cap \{Y_i \in I_i^{>0}\}) = \int_{y_i \in I_i^{>0}} \mathbb{P}(A_i^{y_i}) \geq p \cdot \mathbb{P}(Y_i \in I_i^0). \quad (4)$$

Let $\Pi_{i=1}^l \{x_i\}$ be the event $X_i = x_i$, $\Pi_{i=1}^l \{y_i\}$ be the event $Y_i = y_i$ and $\Pi_{i=1}^l (\{x_i\} \times \{y_i\})$ be the event $X_i = x_i, Y_i = y_i$ for all $i \in [l]$. We declare that $\Pi_{i=1}^l I_i^0 \subseteq \bigcup_{i \in [m] \setminus [l]} A_i$. This is by the following two observations:

1. $\bigcup_{i \in [m] \setminus [l]} A_i$ is independent of X_1, \dots, X_l . Thus, if $\Pi_{i=1}^l (\{x_i\} \times \{y_i\}) \subseteq \bigcup_{i \in [m] \setminus [l]} A_i$ for some $\{x_i\}_{i \in [l]}$, we have $\Pi_{i=1}^l \{y_i\} \subseteq \bigcup_{i \in [m] \setminus [l]} A_i$.

2. For any y_1, \dots, y_l where $y_i \in I_i^0$ for all $i \in l$, we have $\prod_{i=1}^l (\{x_i\} \times \{y_i\}) \subseteq \bigcup_{i \in [m] \setminus [l]} A_i$ for some $\{x_i\}_{i \in [l]}$. This is because $A_i^{y_i} \subseteq \{a\} \times \{b\}$ for all i , and it is easy to see we can always find $\{x_i\}_{i \in [l]}$ such that the event $\prod_{i=1}^l \{x_i\}$ has no intersection with $A_k^{y_k}$ for any $k \in [l]$, which implies $\prod_{i=1}^l (\{x_i\} \times \{y_i\}) \cap A_k = \emptyset$, and further $\prod_{i=1}^l (\{x_i\} \times \{y_i\}) \cap (\bigcup_{i \in [l]} A_i) = \emptyset$. Then, we have $\prod_{i=1}^l (\{x_i\} \times \{y_i\}) \subseteq \bigcup_{i \in [m] \setminus [l]} A_i$ by $\mathbb{P}(\bigcup_{A \in \mathcal{A}} A) = 1$.

According to whether there is some $i \in [l]$ such that $\mathbb{P}(Y_i \in I_i^{>0}) \geq p$, it is divided into two cases.

Case 1: $\forall i \in [l], \mathbb{P}(Y_i \in I_i^{>0}) < p$.

Let A'_i be the event $\{Y_i \in [1-p, 1]\}$ for $i \in [l]$ and A'_i be A_i for other $i \in [m] \setminus [l]$. It is easy to see $\mathbb{P}(\mathcal{A}') = \mathbf{p}$. Meanwhile, we also have $\mathbb{P}(\bigcup_{i \in [m]} A'_i) = 1$. This is by the following two facts: event $\{Y_i \in [1-p, 1]\}$ is exact A'_i for all $i \in [l]$ and the event $\{Y_i < p \text{ for any } i \in [l]\}$ is covered by $\prod_{i=1}^l I_i^0 \subseteq \bigcup_{i \in [m] \setminus [l]} A_i = \bigcup_{i \in [m] \setminus [l]} A'_i$.

Note that $\mathcal{A}' \sim G'_B$, where G'_B is obtained from G_B by deleting the the edges on C_l , i.e., the edges between A_1, A_2, \dots, A_l and X_1, X_2, \dots, X_l . Thus, we have $p \geq P_V(G'_B)$ since $\mathbb{P}(\bigcup_{i \in [m]} A'_i) = 1$ and $\mathbb{P}(A'_i) = p$ for any $i \in [m]$. Meanwhile, we also have $p \leq P_V(G'_B)$ by noting that $\mathcal{V}\mathcal{I}(G_B) \subseteq \mathcal{V}\mathcal{I}(G'_B)$ and $p = P_V(G'_B)$. In summary, we have $p = P_V(G'_B)$ and $\mathbf{p} \in \partial_v(G'_B)$. Since $|\mathcal{C}(G'_B)| < |\mathcal{C}(G_B)| = N + 1$, by the induction hypothesis, there is another standard event set $\mathcal{A}'' \sim G'_B$ such that $\mathbb{P}(\mathcal{A}'') = \mathbf{p}$, $\mathbb{P}(\bigcup_{i \in [m]} A''_i) = 1$ and for every cyclic graph $C \in \mathcal{C}(G'_B)$, there are three events A''_i, A''_j, A''_k on C where A''_j is neighboring to A''_i and A''_k and satisfying $\mathbb{P}(A''_j \cap (A''_i \cup A''_k)) \geq p^3$.

Observe that \mathcal{A}'' is also a standard event of G_B and let C be any cyclic graph in $\mathcal{C}(G_B) \setminus \mathcal{C}(G'_B)$. Note that at least one of X_1, \dots, X_l is on C , thus C and C_l should share at least two adjacent events, say A''_i and A''_{i+1} . Since A''_i and A''_{i+1} are independent, we have $\mathbb{P}(A''_i \cap A''_{i+1}) = p^2$. Then the conclusion follows.

Case 2: $\mathbb{P}(Y_i \in I_i^{>0}) \geq p$ holds for some $i \in [l]$.

Note that $A_i \cap \{Y_i \in I_i^1\} = \{X_i = a, Y_i \in I_i^1\}$ is independent of the shared variable with A_{i+1} , namely X_{i+1} . Thus, by letting E_1 be $\{X_i = a, Y_i \in I_i^1\}$, we have $\mathbb{P}(E_1 \cap A_{i+1}) = p \cdot \mathbb{P}(E_1)$. Similarly, let $E_2 := A_i \cap \{Y_i \in I_i^2\} = \{X_{i+1} = b, Y_i \in I_i^2\}$ and $E_4 := A_i \cap \{Y_i \in I_i^4\} = \{Y_i \in I_i^4\}$, we can also have $\mathbb{P}(E_2 \cap A_{i-1}) = p \cdot \mathbb{P}(E_2)$ and $\mathbb{P}(E_4 \cap A_{i-1}) = p \cdot \mathbb{P}(E_4)$.

Let $E'_3 = \{X_i = a, Y_i \in I_i^3\}$, $E''_3 = \{X_i = 1-a, X_{i+1} = b, Y_i \in I_i^3\}$ and $E_3 = E'_3 \cup E''_3$, then $E_3 = A_i \cap \{Y_i \in I_i^3\}$. It is easy to verify that $\mathbb{P}(E'_3 \cap A_{i+1}) = p \cdot \mathbb{P}(E'_3)$. In addition, since \mathcal{A} is standard, A_{i-1} and A_i are both X_i -monotone in the opposite directions. Meanwhile, it is easy to verify that E'_3 and A_i are both X_i -monotone in the opposite directions. Thus, A_{i-1} and E''_3 are X_i -monotone in the same direction. Then, we have $\mathbb{P}(E''_3 \cap A_{i-1}) \geq \mathbb{P}(E''_3)\mathbb{P}(A_{i-1}) = p \cdot \mathbb{P}(E''_3)$ according to Lemma 5.6 (c). Therefore, $\mathbb{P}(E_3 \cap (A_{i-1} \cup A_{i+1})) \geq p \cdot (\mathbb{P}(E'_3) + \mathbb{P}(E''_3)) = p \cdot \mathbb{P}(E_3)$. Thus,

$$\mathbb{P}(A_i \cap (A_{i-1} \cup A_{i+1})) \geq \sum_{i=1}^4 \mathbb{P}(E_i \cap (A_{i-1} \cup A_{i+1})) \geq p \cdot \sum_{i=1}^4 \mathbb{P}(E_i) \geq p^2 \cdot \mathbb{P}(Y_i \in I_i^{>0}) \geq p^3,$$

where the third inequality is due to (4). Then the conclusion follows. \square

A nice property about 2-discrete cyclic graph when considering CLLL is that all associated qudits are solitary. Thus, by Theorem 5.5 it is lossless to assume these are classical variables, and the subspaces associated are classical events. So the above theorem can be extended to the commuting case naturally.

5.3 Probability Transfer on Dependency Graph

In this subsection, we introduce an operation, named *probability transfer*, on the dependency graph. Let \mathbf{e}_i be the unit vector (p_1, \dots, p_m) where $p_i = 1$ and $p_j = 0$ for any $j \neq i$. The following lemma gives the element operation of probability transfer and shows that the probability can transfer between two neighbors with an amplification rate $\frac{1-p_i}{p_j}$.

Lemma 5.11. *Let $G_D = ([m], E_D)$ be a dependency graph and \mathbf{p} be a probability vector beyond shearer's bound. Then for any $i \in \Gamma_j$ and $q \leq p_j$, $\mathbf{p}' = \mathbf{p} - q \cdot (\mathbf{e}_j - (1 - p_i)/p_j \cdot \mathbf{e}_i)$ is also beyond shearer's bound.*

Proof. Let $\mathcal{A} \sim G_D$ be an event set with $\mathbb{P}(\mathcal{A}) = \mathbf{p}$ and $\mathbb{P}(\cup_{i \in [m]} A_i) = 1$. We construct a new event set \mathcal{A}' as follows: $A'_j := A_j \setminus B$, $A_i := A_i \cup B$ and $A'_k = A_k$ for the other events. Here B is an event independent of all the events in \mathcal{A} with probability q/p_j . It is easy to verify that $\mathcal{A}' \sim G_D$, $\mathbb{P}(\cup_{i \in [m]} A'_i) = 1$ and $\mathbb{P}(\mathcal{A}') = \mathbf{p}'$. \square

It is easy to see that the probability can also transfer between independent events by performing the element operations in turn along the path between these events.

Lemma 5.12. *Let $G_D = ([m], E_D)$ be a dependency graph and \mathbf{p} be a probability vector beyond shearer's bound. Suppose $i, i_1, i_2, \dots, i_{k-1}, j$ is a shortest path from i to j , and $p_{i_1} = \dots = p_{i_{k-1}} = p_j = p$, then for any $q \leq p$, $\mathbf{p} - q \cdot (\mathbf{e}_j - (\frac{1-p}{p})^{k-1} \cdot \frac{1-p_i}{p} \cdot \mathbf{e}_i)$ is also beyond Shearer's bound.*

Definition 5.7. Given a dependency graph $G_D = ([m], E_D)$, we denote the distance between i and j by $\text{Dis}(i, j)$ (e.g., $\text{Dis}(i, i) = 0$). For any $i \in [m]$, let $\Gamma_i^+(l)$ be $\{j \in [m] : \text{Dis}(i, j) \leq l\}$. Specially, $\Gamma_i^+(0) = \{i\}$ and $\Gamma_i^+(1) = \Gamma_i^+$. Let $\Gamma_i(l) := \Gamma_i^+(l) \setminus \{i\}$, and $\Delta(G_D)$ be the maximum degree of the vertices in G_D . Given any $T \subseteq \Gamma_i^+(l)$, we say T is (i, l) -concentrated if $i \in T \subseteq \Gamma_i^+(l)$ and T contains a shortest path connecting i and j for any $j \in T$.

Given an interaction bipartite graph $G_B = ([m], [n], E)$, we define $\text{Dis}(i, j)$, $\Gamma_i^+(l)$, $\Gamma_i(l)$, $\Delta(G_B)$ and concentrated subset to be those of its dependency graph $G_D(G_B)$.

Given an (i, l) -concentrated set T , the probability can transfer from each $j \in T \setminus \{i\}$ to i .

Theorem 5.13. *Given a dependency graph $G_D = ([m], E_D)$, an (i, l) -concentrated set T and a probability vector \mathbf{p} where $p_j = p$ for any $j \in T$, let T_k be $\{j \in T : \text{Dis}(i, j) = k\}$. For any $q_1 \leq p$ and $q_2 \leq q_1(1 + \sum_{0 < k \leq l} (1 - p + q_1)|T_k|(1 - p)^{k-1}/p^k)^{-1}$, if $\mathbf{p} - q_1 \mathbf{e}_i$ is beyond shearer's bound for G_D , then $\mathbf{p} - \sum_{j \in T} q_2 \mathbf{e}_j$ is also beyond shearer's bound.*

Based on Lemma 5.12, it is not difficult to prove Theorem 5.13. The main idea is to deal with the vertices in $T \setminus \{i\}$ layer by layer, i.e., in the order T_l, T_{l-1}, \dots, T_1 . The detailed proof is omitted here.

Definition 5.8. Given any positive integers d, l and any $0 < q \leq p \leq 1$, let

$$\tau(d, l, p, q) = \frac{q(p^{l+1} - p^l(d-1)(1-p))}{p^{l+1} - p^l(d-1)(1-p) + (1-p+q)d(p^l - (d-1)^l(1-p)^l)}.$$

It is easy to verify that $\tau(\Delta(G_D), l, p, q_1) \leq q_1(1 + \sum_{0 < k \leq l} (1 - p + q_1)|T_k|(1 - p)^{k-1}/p^k)^{-1}$ in the setting of theorem 5.13. Thus, we have the following corollary.

Corollary 5.14. *In the setting of Theorem 5.13, $\mathbf{p} - \sum_{j \in T} q_2 \mathbf{e}_j$ is also beyond shearer's bound for any $q_2 \leq \tau(\Delta(G_D), l, p, q_1)$.*

5.4 Lower Bound for the Gaps between Critical Thresholds

Now, we are ready to prove Theorem 1.8, the main theorem of this section.

Theorem 1.8 . *Given a bipartite graph $G_B = ([m], [n], E)$ and a constant l , if for any $i \in [m]$, there is another $j \in [m]$ on a 2-discrete l_1 -cyclic graph where $\text{Dis}(i, j) \leq l - \lfloor l/2 \rfloor - 2$, then $P_V \geq R_C > R_Q = P_A$, $P_V - R_Q \geq \tau(\Delta(G_B), l, P_V, P_V^3) \geq C \cdot \frac{P_V^{l+3}}{(1-P_V)^{l(\Delta(G_B)-1)^l}}$ and $R_C - R_Q \geq \tau(\Delta(G_B), l, R_C, R_C^3) \geq C \cdot \frac{R_C^{l+3}}{(1-R_C)^{l(\Delta(G_B)-1)^l}}$. Here C is an absolute constant (say $\frac{1}{25}$ is enough).*

Proof. Let $p = P_V(G_B)$ and $\mathbf{p} = (p, \dots, p)$. By Theorem 5.10, there is a standard event set \mathcal{A} of G_B such that $\mathbb{P}(\mathcal{A}) = \mathbf{p}$, $\mathbb{P}(\bigcup_{i \in [m]} A_i) = 1$ and for every 2-discrete cyclic graph $C \in \mathcal{C}(G_B)$, there are three events $A_{i_1}, A_{i_2}, A_{i_3}$ on C such that A_{i_2} is neighboring to A_{i_1} and A_{i_3} and $\mathbb{P}(A_{i_2} \cap (A_{i_1} \cup A_{i_3})) \geq p^3$. Let S_C be the set of cuttings $\{(i_2, i_1), (i_2, i_3)\}$.

For any compatible set of cuttings S , let $S_1 := \{k', (k', i') \in S\}$ and $S_2 := \{i', (k', i') \in S\}$. Let C_{l_1} be any l_1 -cyclic graph in $\mathcal{C}(G_B)$. By the definition of compatible set, we can construct a compatible set S such that either $S_{C_{l_1}} \subseteq S$, or there exists $i' \in S_2$ such that the distance between i' and some $i \in C_{l_1}$ is no larger than 1. It is easy to verify that in both cases, there exists $k' \in S_1$ such that the distance between k' and any left vertex on C_{l_1} is no larger than $\lfloor l/2 \rfloor + 2$. Combined with the condition of the theorem, we have for any $i \in [m]$, the distance between i and the closest $k' \in S_1$ is no larger than l . For a given $i \in [m]$, there may be more than one closest vertex in S . We break the tie by letting k' be the one with minimum index. And we let $F(i) := k'$, then $F^{-1}(k') = \{i : F(i) = k'\}$ is (k', l) -concentrated.

Because $\mathbf{p} \in \partial_v(G_B)$, from Theorem 5.4, we have there is a standard event set \mathcal{A} of G_B such that $\mathbb{P}(\mathcal{A}) = \mathbf{p}$ and $\mathbb{P}(\bigcup_{A \in \mathcal{A}} A) = 1$. Let \mathcal{A}' be the resulted event set by applying the set of cuttings S on \mathcal{A} . By Theorem 5.9, we have $\mathbb{P}(\bigcup_{A' \in \mathcal{A}'} A') = \mathbb{P}(\bigcup_{A \in \mathcal{A}} A) = 1$, and $G_D(G_B)$ is the lopsidedependency graph of event set \mathcal{A}' . Combining these two facts with Theorem 5.1, we have $\mathbb{P}(\mathcal{A}') \leq \mathbf{p} - \sum_{k' \in S_1} p^3 \mathbf{e}_{k'}$ is beyond Shearer's bound. Finally, we transfer the probabilities on vertices in $F^{-1}(k')$ to k' for all $k' \in S_1$ in parallel. By Corollary 5.14, we have $\mathbf{p} - \sum_{j \in [m]} \tau(\Delta(G_D), l, p, p^3) \mathbf{e}_j$ is also beyond Shearer's bound.

Following almost the same steps, we can also prove the result for R_C . \square

Theorem 1.8 shows that for any finite graph (i.e., $m < +\infty$) containing a 2-discrete cyclic graph, P_V and R_C are exactly larger than $P_A = R_Q$. Meanwhile, it also gives lower bounds for P_V and R_C , which are exactly large than R_Q . An interesting corollary of Theorem 1.8 is about cycles, which has received considerable attention in the LLL literature [24, 28]: For any l -cyclic graph, we have $R_C = P_V$ and $R_Q = P_A$ and $P_V - P_A \geq \tau(2, l, p, p^3) \geq \frac{1}{25} p^3 \cdot (\frac{p}{1-p})^l$. Indeed, we can obtain better bound by exploiting the specify structure of cycles.

Corollary 1.9 . *For any l -cyclic graph, we have $R_C = P_V$, $R_Q = P_A$ and $P_V - P_A \geq \tau(2, \lfloor \frac{l-1}{2} \rfloor, p, \frac{p^2}{2}) \geq \frac{1}{50} p^2 \cdot (\frac{p}{1-p})^{\lfloor \frac{l-1}{2} \rfloor}$.*

Proof. Given an l -cyclic graph $C_l = ([l], [l], E_l)$, let $p = P_V$ and $\mathbf{p} = (p, \dots, p)$. By Theorem 4 in [24], there is a standard event set \mathcal{A} of C_l such that $\mathbb{P}(\mathcal{A}) = \mathbf{p}$, $\mathbb{P}(\bigcup_{i \in [l]} A_i) = 1$ and two adjacent events, say A_1 and A_2 , are independent. In other words, A_1 and A_2 are independent of X_2 and $\mathbb{P}(A_1 \cap A_2) = p^2$. Without loss of generality, we assume that $X_2 \in \{0, 1\}$ and $\mathbb{P}(X_2 = 0) = \mathbb{P}(X_2 = 1) = 0.5$. We construct a new event set \mathcal{A}' as follows: $A'_1 = A_1 \setminus (A_1 \cap A_2 \cap \{X_2 = 1\})$, $A'_2 = A_2 \setminus (A_1 \cap A_2 \cap \{X_2 = 0\})$, and $A'_i = A_i$ for all $i \geq 2$. Since $A'_1 \cup A'_2 = A_1 \cup A_2$, we have $\mathbb{P}(\bigcup_{i \in [l]} A'_i) = 1$. In addition, note

that X_2 is only shared by A'_1 and A'_2 , it is not hard to verify that $G_D(C_l)$ is a lopsidedependency graph of \mathcal{A}' . Hence, $\mathbf{p}' = \mathbf{p} - \frac{p^2}{2}(\mathbf{e}_1 + \mathbf{e}_2)$ is still beyond Shearer's bound. For any $i \in [l] \setminus \{1, 2\}$, we let i in T_1 or T_2 according to whether $\text{Dis}(i, 2) \geq \text{Dis}(i, 1)$ or not. Then for each T_b where $b \in \{1, 2\}$, we transfer the probabilities on vertices in T_b to b . By Corollary 5.14, we have $\mathbf{p} - \tau(2, \lfloor \frac{l-1}{2} \rfloor, p, \frac{p^2}{2})$ also beyond Shearer's bound. \square

5.5 Application to Infinite Graphs

Given a dependency graph G_D , it naturally defines an event-variable graph $G_B(G_D)$ as follows. Regard each edge of G_D as a variable and each vertex as an event. An event A depends on a variable X if and only if the vertex corresponding to A is an endpoint of the edge corresponding to X . In this subsection, we investigate the critical thresholds of $G_B(G_D)$ for a given dependency graph G_D . We obtain lower bounds for the critical thresholds of periodic Euclidean graphs, which include many lattices.

Here, G_D is called a Euclidean graph if the vertices are points in Euclidean space, and the edges are line segments connecting pairs of vertices. In other words, G_D is embedded in some Euclidean space. A Euclidean graph is periodic if there exists a basis of that Euclidean space whose corresponding translations induce symmetries of that graph (i.e., application of any such translation to the graph embedded in the Euclidean space leaves the graph unchanged). Given a periodic Euclidean graph G_D , we say a finite graph G_U is a translational unit of G_D if G_D can be viewed as the union of periodic translations of G_U . For example, the square lattice is a periodic graph with the cycle of length 4 as a translational unit.

Periodic graphs have been extensively studied in natural science and engineering, particularly of three-dimensional crystal nets to crystal engineering, crystal prediction (design), and modeling crystal behavior [8, 26, 43].

The following result is a direct application of Theorem 1.8.

Theorem 1.10 . *Let G_D be a periodic graph. If G_D is a tree, then $P_A = R_Q = P_V = R_C$. Otherwise, G_D has a translational unit G_U which has a induced subgraph as a cycle, and we have $P_A = R_Q$, $P_V = R_C$ and $P_V - R_Q \geq \tau(\Delta(G_D), l, P_V, P_V^3) \geq \frac{1}{25} \cdot \frac{P_V^{l+3}}{(1-P_V)^{l(\Delta(G_D)-1)l}}$, where l is the number of vertices of G_U .*

Proof. If G_D is a tree, the corresponding G_B is also a tree. By Theorem 4.15, we have $P_A = P_V = R_C = R_Q$.

Otherwise, any vertex i in G_D is included in a translational unit G_U . Let C be the cycle contained in G_U and the length of C is l_1 . It is easy to see that there exists j in C such that $\text{Dis}(i, j) + \lfloor l_1/2 \rfloor + 2 \leq l$, thus the conclusion follows immediately by applying Theorem 1.8. \square

An important application of Theorem 1.10 is to lower bound the critical thresholds of lattices. The critical thresholds of many infinite lattices have been studied extensively in physics, including the research on hard-core singularity in the statistical mechanical literature [25, 46] and that on local Hamiltonians in quantum physics [36, 39]. Many of these lattices studied in the literature are periodic graphs. Thus, we can lower bound the critical thresholds on these lattices with Theorem 1.10. We illustrate this with the square lattice, \mathbb{Z}^2 . Note that $\Delta(G_D) = 4$, the cycle of length 4 is a translational unit, and $P_A(\mathbb{Z}^2) \geq 0.11933888188(1)$ [46]. Then $R_C(\mathbb{Z}^2) = P_V(\mathbb{Z}^2) \geq P_A(\mathbb{Z}^2) + \tau(4, 4, P_V(\mathbb{Z}^2), P_V^3(\mathbb{Z}^2)) \geq P_A(\mathbb{Z}^2) + \tau(4, 4, P_A(\mathbb{Z}^2), P_A^3(\mathbb{Z}^2)) \geq P_A + 5.057 \times 10^{-9} \geq 0.11933888693$.

The lower bound in Theorem 1.10 can be improved further by taking the specific structures of lattices into consideration. The key idea is to design the set $F^{-1}(k')$ in the proof of Theorem 1.8 carefully.

Let $G_D = ([m], E_D)$ be a lattice such that any vertex in $G_B(G_D)$ is on a 2-discrete cyclic graph. By Theorem 5.10, it is easy to verify that on any $C \in \mathcal{C}(G_B)$, there is some cutting (k', i') such that $\mathbb{P}(A_{k'} \cap A_{i'}) \geq p^3/2$. In the following, we only consider the set of cuttings $T = \{(k', i') : \mathbb{P}(A_{k'} \cap A_{i'}) \geq p^3/2\}$. For any k' , let $i'(k')$ be the minimum one in $\{i' : (k', i') \in T\}$ and set $\mathcal{C}(k')$ as $\{C \in \mathcal{C}(G_B) : \text{there is some } i \neq k' \text{ on } C \text{ such that } \text{Dis}(i, i'(k')) \leq 1 \text{ or } k' \text{ is on } C\}$. It is easy to verify that for each $C' \in \mathcal{C}(G_B) \setminus \mathcal{C}(k')$, there is another cutting $(t', l') \in T$ compatible with $(k', i'(k'))$ where $t' \neq k'$. Thus, we can design a compatible set of cuttings S in a greedy way. In the beginning, $S = \emptyset$. We choose a cutting $(k', i'(k')) \in T$ into S and delete the cuttings on the cyclic graphs in $\mathcal{C}(k')$ from T . Then, we choose another cutting into S and delete corresponding cuttings from T again, until T is \emptyset . In the end, we can set $F^{-1}(k')$ as $\{i \in [m] : i \text{ is on } \mathcal{C}(k')\}$ and it is easy to verify that $\bigcup_{k' \in S_1} F^{-1}(k') = [m]$.

For square lattice, we have for any k' , $|\{i : \text{Dis}(i, k') = 1\} \cap F^{-1}(k')| \leq 4$, $|\{i : \text{Dis}(i, k') = 2\} \cap F^{-1}(k')| \leq 7$, $|\{i : \text{Dis}(i, k') = 3\} \cap F^{-1}(k')| \leq 5$ and $|\{i : \text{Dis}(i, k') = 4\} \cap F^{-1}(k')| \leq 4$. Thus, by Theorem 5.13, it is easy to verify that $P_V - P_A \geq 5.943 \times 10^{-8}$.

We also apply our theorem to the hexagonal, triangular and simple cubic lattices. For hexagonal lattice, we have for any k' , $|\{i : \text{Dis}(i, k') = 1\} \cap F^{-1}(k')| \leq 3$, $|\{i : \text{Dis}(i, k') = 2\} \cap F^{-1}(k')| \leq 6$, $|\{i : \text{Dis}(i, k') = 3\} \cap F^{-1}(k')| \leq 5$, $|\{i : \text{Dis}(i, k') = 4\} \cap F^{-1}(k')| \leq 5$ and $|\{i : \text{Dis}(i, k') = 5\} \cap F^{-1}(k')| \leq 2$. Thus, by Theorem 5.13, it is easy to verify that $P_V - P_A \geq 1.211 \times 10^{-7}$.

For triangular lattice, we have for any k' , $|\{i : \text{Dis}(i, k') = 1\} \cap F^{-1}(k')| \leq 6$, $|\{i : \text{Dis}(i, k') = 2\} \cap F^{-1}(k')| \leq 7$ and $|\{i : \text{Dis}(i, k') = 3\} \cap F^{-1}(k')| \leq 5$. Thus, by Theorem 5.13, it is easy to verify that $P_V - P_A \geq 6.199 \times 10^{-8}$.

For simple cubic lattice, we only consider the cuttings in the horizontal planes and omit the cuttings (k', i') where the height of k' and that of i' are different. Thus, for the horizontal plane where k' is on, we have $|\{i : \text{Dis}(i, k') = 1\} \cap F^{-1}(k')| \leq 4$, $|\{i : \text{Dis}(i, k') = 2\} \cap F^{-1}(k')| \leq 7$, $|\{i : \text{Dis}(i, k') = 3\} \cap F^{-1}(k')| \leq 5$ and $|\{i : \text{Dis}(i, k') = 4\} \cap F^{-1}(k')| \leq 4$. For the horizontal plane above k' and that below k' , we have $|\{i : \text{Dis}(i, k') = 1\} \cap F^{-1}(k')| \leq 1$, $|\{i : \text{Dis}(i, k') = 2\} \cap F^{-1}(k')| \leq 3$, $|\{i : \text{Dis}(i, k') = 3\} \cap F^{-1}(k')| \leq 3$ and $|\{i : \text{Dis}(i, k') = 4\} \cap F^{-1}(k')| \leq 2$. In summary, we have $|\{i : \text{Dis}(i, k') = 1\} \cap F^{-1}(k')| \leq 6$, $|\{i : \text{Dis}(i, k') = 2\} \cap F^{-1}(k')| \leq 13$, $|\{i : \text{Dis}(i, k') = 3\} \cap F^{-1}(k')| \leq 11$ and $|\{i : \text{Dis}(i, k') = 4\} \cap F^{-1}(k')| \leq 8$. Thus, by Theorem 5.13, it is easy to verify that $P_V - P_A \geq 9.533 \times 10^{-10}$. We have summarized our results on lattices in Table 1.

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A Theorem for Gap Decision

We study whether Shearer's bound is tight for CLLL on a given interaction bipartite graph in this appendix. Our main result, namely Theorem 4.9, is a sufficient and necessary condition for deciding whether Shearer's bound remains tight for CLLL on a given interaction bipartite graph, which is an extension of [24, Theorem 5] for VLLL. It bridges gaplessness and exclusiveness in the interior. Though this theorem seems similar to Theorem 5 in [24], the proof is very different. The proof in [24] relies on the exclusive cylinder set on the boundary, which connects gaplessness with exclusiveness naturally. The existence of such exclusive cylinder set is ensured by Theorem 3 [24], the key idea of which is that the discreteness degree of each variable is bounded by the number of events related to this variable. However, for CLLL there is no such subspace set on the boundary if the relative dimension on the boundary is irrational. Even if the relative dimensions are rational, it is still challenging to bound the discreteness degree of subspaces because of the possible entanglement. Thus, we need new techniques to connect gaplessness with exclusiveness. Roughly speaking, in our proof, we first get a commuting subspace set, the relative dimensions of which exceeds the boundary. Then we adapt it to be exclusive by slicing the subspaces and discarding some slices. The main techniques used are a probability tool shown in Lemma A.3 and Lemma 4.6.

Here are some properties of classical exclusive event sets, which will be used.

Lemma A.1 (Theorem 1 in [44]). *Given G_D and $\mathbf{p} \in \mathcal{I}(G_D) \cup \partial(G_D)$. Among all event sets $\mathcal{A} \sim G_D$ with $\mathbb{P}(\mathcal{A}) = \mathbf{p}$, there is an exclusive one such that $\mathbb{P}(\cup_{A \in \mathcal{A}} A)$ is maximized.*

Lemma A.2 (Lemma 29 in [24]). *Suppose that G_D is a dependency graph of event sets \mathcal{A} and \mathcal{B} , $\mathbb{P}(\mathcal{A}) = \mathbb{P}(\mathcal{B})$, and \mathcal{B} is exclusive. Then $\mathbb{P}(\cup_{A \in \mathcal{A}} A) \leq \mathbb{P}(\cup_{B \in \mathcal{B}} B)$, and the equality holds if and only if \mathcal{A} is exclusive.*

By Lemma A.1 and Lemma A.2, we have that for any event set $\mathcal{A} \sim G_D$ where $\mathbb{P}(\mathcal{A}) \in \mathcal{I}(G_D) \cup \partial(G_D)$ and any i, j where $i \in \Gamma_j$, if $\mathbb{P}(A_i \cap A_j) > 0$, then $\mathbb{P}(\cap_{A \in \mathcal{A}} \bar{A}) > 0$. The following lemma, namely Lemma A.3, further shows that $\mathbb{P}(\cap_{A \in \mathcal{A}} \bar{A})$ can be lower bounded, and can be viewed as a quantitative version of Lemma A.2.

Define $I(G_D, \mathbf{p}, k) := \min\{I(G_D(V'), (p_v : v \in V')) : |V'| = k\}$. Let p_{min} be the minimum element in $\mathbf{p} = (p_1, p_2, \dots, p_m)$. If $\mathbf{p} \in \mathcal{I}(G_D) \cup \partial(G_D)$ and $t \leq m - 2$, we can define

$$\mathbb{F}(G_D, \mathbf{p}, t) =: \begin{cases} p_{min}^t \prod_{k=1}^t \frac{I(G_D, \mathbf{p}, k)}{(m-1-k)} & \text{if } \mathbf{p} \in \partial(G_D), \\ I(G_D, \mathbf{p}) & \text{if } \mathbf{p} \in \mathcal{I}(G_D). \end{cases} \quad (5)$$

It is not hard to see $\mathbb{F}(G_D, \mathbf{p}, t) > 0$, $\mathbb{F}(G, \mathbf{p}, t') \leq \mathbb{F}(G, \mathbf{p}, t)$ for $t' \geq t$, and $\mathbb{F}(G_D, \mathbf{p}', t) \leq \mathbb{F}(G_D, \mathbf{p}, t)$ for any $\mathbf{p}' \geq \mathbf{p}$.

Lemma A.3. *Given a dependency graph $G_D = ([m], E_D)$, a vector $\mathbf{p} \in \mathcal{I}(G_D) \cup \partial(G_D)$. For any event set $\mathcal{A} \sim G_D$ where $\mathbb{P}(\mathcal{A}) = \mathbf{p}$ and any i, j where $i \in \Gamma_j$, we have $\mathbb{P}(\cap_{A \in \mathcal{A}} \bar{A}) \geq \mathbb{P}(A_i \cap A_j) \mathbb{F}(G, \mathbf{p}, m - 2)$.*

Proof. Let $\mathcal{A} = \{A_1, \dots, A_m\}$ be a set of events conforms with G_D where $\mathbb{P}(\mathcal{A}) = \mathbf{p} = (p_1, \dots, p_m)$. From $\mathbf{p} \in \mathcal{I}(G_D) \cup \partial(G_D)$ and Lemma A.1, there exists an exclusive event set $\mathcal{B} \sim G_D$ where $\mathbb{P}(\mathcal{B}) = \mathbf{p}$. We assume $\mathbb{P}(A_i \cap A_j) > 0$, since the case $\mathbb{P}(A_i \cap A_j) = 0$ holds trivially. Further, we can assume $\mathbf{p} \in \partial(G_D)$, since otherwise $\mathbf{p} \in \mathcal{I}(G_D)$, then due to Theorem 1.2, we have $\mathbb{P}(\cap_{A \in \mathcal{A}} \bar{A}) \geq I(G, \mathbf{p}) = \mathbb{F}(G, \mathbf{p}, m - 2) \geq \mathbb{P}(A_i \cap A_j) \mathbb{F}(G, \mathbf{p}, m - 2)$.

Let's borrow the notation from the proof of Theorem 1 in [44]. For any $S \subseteq [m]$, define $\alpha(S) = \mathbb{P}(\cap_{i \in S} \overline{A_i})$ and $\beta(S) = \mathbb{P}(\cap_{i \in S} \overline{B_i})$. We first review some useful properties of $\alpha(S)$ and $\beta(S)$. Note that $\alpha(S)/\beta(S)$ monotonically increases as $|S|$ increases provided $\beta(S) \neq 0$. This can be proved by induction on $|S|$. The base cases holds since $\alpha(\emptyset) = \beta(\emptyset)$ and $\alpha(S) = \beta(S)$ for any singleton S . For induction, given $S_1 \subset [m]$ and $j \in [m] \setminus S_1$, let $S_2 = S_1 \cup \{j\}$, $T_2 = S_1 \cap \Gamma_j$, and $T_1 = S_1 \setminus T_2$. We have

$$\frac{\alpha(S_2)}{\beta(S_2)} - \frac{\alpha(S_1)}{\beta(S_1)} \geq \frac{\alpha(S_1) - p_j \alpha(T_1)}{\beta(S_1) - p_j \beta(T_1)} - \frac{\alpha(S_1)}{\beta(S_1)} = \frac{p_j \beta(T_1)}{\beta(S_1) - p_j \beta(T_1)} \left[\frac{\alpha(S_1)}{\beta(S_1)} - \frac{\alpha(T_1)}{\beta(T_1)} \right] \geq 0. \quad (6)$$

The last inequality is by induction. The first inequality holds because

$$\begin{aligned} \alpha(S_2) &= \mathbb{P}(\cap_{i \in S_2} \overline{A_i}) = \mathbb{P}(\cap_{i \in S_1} \overline{A_i}) - \mathbb{P}(\cap_{i \in S_1} \overline{A_i} \cap A_j) \\ &= \mathbb{P}(\cap_{i \in S_1} \overline{A_i}) - \mathbb{P}(\cap_{i \in T_1} \overline{A_i} \cap A_j) + \mathbb{P}(\cap_{i \in T_1} \overline{A_i} \cap A_j \cap (\cup_{i \in T_2} A_i)) \\ &\geq \mathbb{P}(\cap_{i \in S_1} \overline{A_i}) - \mathbb{P}(\cap_{i \in T_1} \overline{A_i} \cap A_j) = \alpha(S_1) - p_j \alpha(T_1). \end{aligned} \quad (7)$$

and by a similarly formula we also have $\beta(S_2) = \beta(S_1) - p_j \beta(T_1)$ because \mathcal{B} is exclusive and then $\mathbb{P}(\cap_{i \in T_1} \overline{B_i} \cap B_j \cap (\cup_{i \in T_2} B_i)) = 0$. Hence, $\alpha(S)/\beta(S)$ is increasing.

Now we return to the proof of the lemma. Let $S_2 = [m]$, $S_1 = S_2 \setminus \{j\}$, $T_1 = S_1 \setminus \Gamma_j$, $T_2 = S_1 \setminus T_1 = \Gamma_j$. Since $\beta(S_1) > 0$ and $\beta(S_2) = 0$, $\alpha(S_2) = \alpha(S_2) - \frac{\alpha(S_1)}{\beta(S_1)} \beta(S_2)$. Moreover, since G_D is connected, $|T_1| \leq m - 2$, we have $\mathbb{P}(A_j \cap (\cup_{i \in T_2} A_i)) \mathbb{F}(G_D, \mathbf{p}, |T_1|) \geq \mathbb{P}(A_i \cap A_j) \mathbb{F}(G_D, \mathbf{p}, m - 2)$. Therefore, to prove $\mathbb{P}(\cap_{A \in \mathcal{A}} \overline{A}) \geq \mathbb{P}(A_i \cap A_j) \mathbb{F}(G_D, \mathbf{p}, m - 2)$, it suffices to show $\alpha(S_2) - \frac{\alpha(S_1)}{\beta(S_1)} \beta(S_2) \geq \mathbb{P}(A_j \cap (\cup_{i \in T_2} A_i)) \mathbb{F}(G_D, \mathbf{p}, |T_1|)$.

Claim. For any $S_2 \subseteq [m]$ and $j \in S_2$, let $S_1 = S_2 \setminus \{j\}$, $T_1 = S_1 \setminus \Gamma_j$, $T_2 = S_1 \setminus T_1$. Then $\alpha(S_2) - \frac{\alpha(S_1)}{\beta(S_1)} \beta(S_2) \geq \mathbb{P}(A_j \cap (\cup_{i \in T_2} A_i)) \mathbb{F}(G_D, \mathbf{p}, |T_1|)$.

Proof of the claim. The following form of $\alpha(S_2) - \frac{\alpha(S_1)}{\beta(S_1)} \beta(S_2)$ will be used:

$$\begin{aligned} \alpha(S_2) - \frac{\alpha(S_1)}{\beta(S_1)} \beta(S_2) &= \alpha(S_2) - \frac{\alpha(S_1)}{\beta(S_1)} (\beta(S_1) - p_j \beta(T_1)) \\ &= \alpha(S_2) - \alpha(S_1) + p_j \alpha(T_1) + p_j \beta(T_1) \left(\frac{\alpha(S_1)}{\beta(S_1)} - \frac{\alpha(T_1)}{\beta(T_1)} \right). \end{aligned} \quad (8)$$

The proof of this claim is by induction on $|T_1|$.

Basis: $T_1 = \emptyset$. We have

$$\begin{aligned} \alpha(S_2) - \frac{\alpha(S_1)}{\beta(S_1)} \beta(S_2) &\geq \alpha(S_2) - \alpha(S_1) + p_j \alpha(T_1) \\ &= \mathbb{P}(\cap_{i \in T_1} \overline{A_i} \cap A_j \cap (\cup_{i \in T_2} A_i)) = \mathbb{P}(A_j \cap (\cup_{i \in T_2} A_i)). \end{aligned}$$

The first inequality is due to formula (8) and the monotonicity of $\alpha(S)/\beta(S)$, the first equality is due to formula (7), and the second equality is due to $T_1 = \emptyset$. Note that $\mathbb{F}(G_D, \mathbf{p}, 0) = 1$, the claim holds for this case.

Hypothesis: The claim holds if $|T_1| < t$.

Induction: Suppose $|T_1| = t$. Since the case $T_2 = \emptyset$ is trivial, we assume $T_2 \neq \emptyset$. By the union bound there is $j' \in T_2$ such that $\mathbb{P}(A_j \cap A_{j'}) \geq \mathbb{P}(A_j \cap (\cup_{i \in T_2} A_i)) / |T_2| \geq \mathbb{P}(A_j \cap (\cup_{i \in T_2} A_i)) / (m - 1 - |T_1|)$, the last inequality is because $|T_2| \leq m - 1 - |T_1|$.

If $T_1 \cap \Gamma_{j'} = \emptyset$, then

$$\begin{aligned}
\alpha(S_2) - \frac{\alpha(S_1)}{\beta(S_1)}\beta(S_2) &\geq \alpha(S_2) - \alpha(S_1) + p_j\alpha(T_1) && \text{by (8)} \\
&= \mathbb{P}(\cap_{i \in T_1} \overline{A_i} \cap A_j \cap (\cup_{i \in T_2} A_i)) && \text{by (7)} \\
&\geq \mathbb{P}(\cap_{i \in T_1} \overline{A_i} \cap A_j \cap A_{j'}) \\
&= \mathbb{P}(\cap_{i \in T_1} \overline{A_i})\mathbb{P}(A_j \cap A_{j'}) && \text{by } T_1 \cap \Gamma_{j'} = \emptyset \\
&\geq I(G_D, \mathbf{p}, |T_1|) \frac{\mathbb{P}(A_j \cap (\cup_{i \in T_2} A_i))}{n-1-|T_1|} && \text{by def. of } I(G_D, \mathbf{p}, t) \\
&\geq \mathbb{F}(G_D, \mathbf{p}, |T_1|)\mathbb{P}(A_j \cap (\cup_{i \in T_2} A_i)). && \text{by def. of } \mathbb{F}(G_D, \mathbf{p}, t)
\end{aligned} \tag{9}$$

Otherwise, $T_1 \cap \Gamma_{j'} \neq \emptyset$, then

$$\begin{aligned}
&\mathbb{P}(A_{j'} \cap (\cup_{i \in T_1 \cap \Gamma_{j'}} A_i)) + \mathbb{P}(\cap_{i \in T_1} \overline{A_i} \cap A_j \cap (\cup_{i \in T_2} A_i)) \\
&\geq \mathbb{P}(A_{j'} \cap (\cup_{i \in T_1 \cap \Gamma_{j'}} A_i)) + \mathbb{P}(\cap_{i \in T_1} \overline{A_i} \cap A_j \cap A_{j'}) \\
&\geq \mathbb{P}(A_{j'} \cap (\cup_{i \in T_1 \cap \Gamma_{j'}} A_i) \cap A_j \cap_{k \in T_1 \setminus \Gamma_{j'}} \overline{A_k}) + \mathbb{P}(\cap_{i \in T_1} \overline{A_i} \cap A_j \cap A_{j'}) \\
&= \mathbb{P}(A_j \cap A_{j'} \cap_{k \in T_1 \setminus \Gamma_{j'}} \overline{A_k}) \\
&= \mathbb{P}(A_j \cap A_{j'})\mathbb{P}(\cap_{k \in T_1 \setminus \Gamma_{j'}} \overline{A_k}),
\end{aligned} \tag{10}$$

we have

$$\begin{aligned}
&\mathbb{P}(A_{j'} \cap (\cup_{i \in T_1 \cap \Gamma_{j'}} A_i)) \\
&\geq \mathbb{P}(A_j \cap A_{j'})\mathbb{P}(\cap_{k \in T_1 \setminus \Gamma_{j'}} \overline{A_k}) - \mathbb{P}(\cap_{i \in T_1} \overline{A_i} \cap A_j \cap (\cup_{i \in T_2} A_i)) \\
&= \mathbb{P}(A_j \cap A_{j'})\mathbb{P}(\cap_{k \in T_1 \setminus \Gamma_{j'}} \overline{A_k}) - (\alpha(S_2) - \alpha(S_1) + p_j\alpha(T_1)).
\end{aligned} \tag{11}$$

The last equality is due to formula (7). Let $S'_2 \triangleq T_1 \cup \{j'\}$, $S'_1 \triangleq T_1$, $T'_1 \triangleq S'_1 \setminus \Gamma_{j'}$, $T'_2 \triangleq S'_1 \setminus T'_1 = T_1 \cap \Gamma_{j'}$. Thus,

$$\begin{aligned}
\frac{\alpha(S_1)}{\beta(S_1)} - \frac{\alpha(T_1)}{\beta(T_1)} &\geq \frac{\alpha(S'_2)}{\beta(S'_2)} - \frac{\alpha(S'_1)}{\beta(S'_1)} = \frac{1}{\beta(S'_2)} (\alpha(S'_2) - \frac{\beta(S'_2)\alpha(S'_1)}{\beta(S'_1)}) \\
&\geq \frac{1}{\beta(S'_2)} \mathbb{P}(A_{j'} \cap (\cup_{i \in T'_2} A_i))\mathbb{F}(G_D, \mathbf{p}, |T'_1|),
\end{aligned} \tag{12}$$

the first inequality is since $S'_1 = T_1$, $S'_2 \subseteq S_1$ and the monotonicity of $\frac{\alpha(S)}{\beta(S)}$. The last inequality is by applying the induction hypothesis to T'_1 . Therefore, we have

$$\begin{aligned}
&\alpha(S_2) - \frac{\alpha(S_1)}{\beta(S_1)}\beta(S_2) \\
&= \alpha(S_2) - \alpha(S_1) + p_j\alpha(T_1) + p_j\beta(T_1) \left(\frac{\alpha(S_1)}{\beta(S_1)} - \frac{\alpha(T_1)}{\beta(T_1)} \right) && \text{by (8)} \\
&\geq \alpha(S_2) - \alpha(S_1) + p_j\alpha(T_1) + p_j\beta(S'_2) \left(\frac{\alpha(S_1)}{\beta(S_1)} - \frac{\alpha(T_1)}{\beta(T_1)} \right) && \text{by } T_1 \subset S'_2 \\
&\geq \alpha(S_2) - \alpha(S_1) + p_j\alpha(T_1) + p_j\mathbb{P}(A_{j'} \cap (\cup_{i \in T'_2} A_i))\mathbb{F}(G_D, \mathbf{p}, |T'_1|) && \text{by (12)} \\
&\geq p_j(\mathbb{P}(A_j \cap A_{j'})\mathbb{P}(\cap_{k \in T_1 \setminus \Gamma_{j'}} \overline{A_k}))\mathbb{F}(G_D, \mathbf{p}, |T'_1|) && \text{by (11)} \\
&\geq p_{\min} I(G, \mathbf{p}, |T_1|)\mathbb{F}(G_D, \mathbf{p}, |T'_1|) \frac{\mathbb{P}(A_j \cap (\cup_{i \in T_2} A_i))}{(m-1-|T_1|)} \\
&\geq \mathbb{P}(A_j \cap (\cup_{i \in T_2} A_i))\mathbb{F}(G_D, \mathbf{p}, |T_1|).
\end{aligned}$$

□

The following property of exclusive subspace sets will also be used.

Lemma A.4. *Given an interaction bipartite graph $G_B = ([m], [n], E_B)$ and a rational vector \mathbf{r} on $(0, 1]$, if there is an exclusive subspace set $\mathcal{V} \sim G_B$ with $R(\mathcal{V}) = \mathbf{r}$, then for any rational $\mathbf{r}' < \mathbf{r}$, there is an exclusive subspace set $\mathcal{V}' \sim G_B$ with $R(\mathcal{V}') = \mathbf{r}'$.*

Proof. W.l.o.g, we can assume that $\mathbf{r}' = (r_1 - \epsilon, r_2, \dots, r_m)$ and \mathcal{H}_1 is related to V_1 . Since \mathbf{r}' and \mathbf{r} are both rational vectors, $\frac{\epsilon}{r_1}$ is rational as well. Suppose $\frac{\epsilon}{r_1} = a/b$ where a and b are integers. Let $\mathcal{H}'_1 = \mathcal{H}_1 \otimes \mathcal{H}_1^c$ where $\dim(\mathcal{H}_1^c) = b$, and $\mathcal{H}'_i = \mathcal{H}_i$ for any $i \geq 2$. Thus the whole vector space is $\mathcal{H}' = \bigotimes_{i=1}^m \mathcal{H}'_i = \bigotimes_{i=1}^m \mathcal{H}_i \otimes \mathcal{H}_1^c$.

We construct the subspace set \mathcal{V}' as follows. Let $V'_1 = V_1 \otimes W$, where W can be any subspace of \mathcal{H}_1^c with dimension $b - a$. For each $i \geq 2$, let $V'_i = V_i \otimes \mathcal{H}_1^c$. It is not difficult to verify that $\mathcal{V}' \sim G_B$, $R(\mathcal{V}') = \mathbf{r}'$, \mathcal{V}' is commuting and exclusive. \square

The following lemma gives the necessary condition of gapless.

Lemma A.5. *Given an interaction bipartite graph $G_B = ([m], [n], E_B)$. For any rational vector $\mathbf{r} \in \mathcal{CI}(G_B) \setminus \mathcal{CD}(G_B)$ on $(0, 1]$ such that G_B is gapless in direction \mathbf{r} , there is an exclusive subspace set $\mathcal{V} \sim G_B$ with $R(\mathcal{V}) = \mathbf{r}$.*

Proof. Let $\mathbf{q} = \lambda \mathbf{r} \in \partial(G_B)$. For any $0 < \epsilon \leq 1$ where $(1 - \epsilon)\mathbf{q}$ is rational, the construction of the exclusive subspace set $\mathcal{V} \sim G_B$ with $R(\mathcal{V}) = (1 - \epsilon)\mathbf{q}$ are as follows. We can assume $m \geq 2$, since the case $m = 1$ is trivial.

Let $\epsilon_1 = \frac{(\epsilon q_{\min})^2}{(3m)^2 \|\mathbf{q}\|_1} \mathbb{F}(G, \mathbf{q}, m - 2)$. Let $\mathbf{q}', \mathbf{q}''$ be rational vectors such that $\mathbf{q} < \mathbf{q}' \leq \phi((1 + \epsilon_1)\mathbf{q})$ and $(1 - \epsilon_1)\mathbf{q} \leq \mathbf{q}'' \leq \mathbf{q}$. Here, $\phi(\mathbf{p}) \in (0, 1]^m$ is the vector whose i -th entry is $\min\{1, p_i\}$ for any i . Note that if $m \geq 2$, then $\mathbf{q} < \mathbf{1}$, thus such \mathbf{q}' always exist.

According to the definition of $\mathcal{CD}(G_B)$, there is a commuting subspace set $\mathcal{V}^{(1)} \sim G_B$ with $R(\mathcal{V}^{(1)}) = \mathbf{q}'$ and $R(\sum_{V_i^{(1)} \in \mathcal{V}^{(1)}} V_i^{(1)}) = 1$, and let $\mathcal{H}^{(1)} = (\mathcal{H}_1^{(1)}, \dots, \mathcal{H}_m^{(1)})$ denote the corresponding qudits.

We construct a commuting subspace set $\mathcal{V}^{(2)} \sim H$ with $R(\mathcal{V}^{(2)}) = \mathbf{q}''$ from $\mathcal{V}^{(1)}$ as follows. Suppose $q''_i/q'_i = a_i/b_i$ where a_i and b_i are integers for $i \in [m]$. Let $\mathcal{H}_i^{(2)} = \mathcal{H}_i^{(1)} \otimes \mathcal{H}_i^c$ where $\dim(\mathcal{H}_i^c) = b_i$, and $V_i^{(2)} = V_i^{(1)} \otimes W_i$ where W_i can be any subspace of \mathcal{H}_i^c with dimension a_i . Let $\mathcal{V}^{(2)} = \{V_1^{(2)}, \dots, V_m^{(2)}\}$. It is not difficult to verify that $\mathcal{V}^{(2)} \sim H$, $\mathbb{P}(\mathcal{V}^{(2)}) = \mathbf{q}''$ and $\mathcal{V}^{(2)}$ is commuting.

Meanwhile, denote the orthogonal complement of W_i in space \mathcal{H}_i^c by \overline{W}_i , we have

$$\begin{aligned} 1 - R(\sum_{V \in \mathcal{V}^{(2)}} V) &= R(\sum_{V \in \mathcal{V}^{(1)}} V) - R(\sum_{V \in \mathcal{V}^{(2)}} V) \\ &= R(\sum_{V \in \mathcal{V}^{(1)}} V \otimes \mathcal{H}_i^c) - R(\sum_{V \in \mathcal{V}^{(2)}} V) \\ &\leq R(\sum_{i \in [m]} V_i^{(1)} \otimes \overline{W}_i) \leq \sum_{i \in [m]} R(V_i^{(1)} \otimes \overline{W}_i) \\ &= \sum_{i \in [m]} R(V_i^{(1)}) R(\overline{W}_i) \leq \sum_{i \in [m]} q'_i (b_i - a_i) / b_i \\ &= \sum_{i \in [m]} (q'_i - q''_i) = \|\mathbf{q}'\|_1 - \|\mathbf{q}''\|_1 \leq 2\epsilon_1 \|\mathbf{q}\|_1. \end{aligned} \quad (13)$$

According to the gaplessness in direction \mathbf{r} , $\mathbf{q}'' \in \mathcal{I}_a(H) \cup \partial_a(G)$. From formula (13) and Lemma A.3, we have for any i, j where $i \in \Gamma_j$,

$$\begin{aligned} R(V_i^{(2)} \cap V_j^{(2)}) &= \mathbb{P}(A_{V_i^{(2)}} \cap A_{V_j^{(2)}}) \\ &\leq \mathbb{P}(\cap_{V_i^{(2)} \in \mathcal{V}^{(2)}} \overline{A_{V_i^{(2)}}}) / \mathbb{F}(G, \mathbf{q}'', n - 2) \\ &\leq \mathbb{P}(\cap_{V_i^{(2)} \in \mathcal{V}^{(2)}} \overline{A_{V_i^{(2)}}}) / \mathbb{F}(G, \mathbf{q}, n - 2) \\ &= (1 - R(\sum_{V_i^{(2)} \in \mathcal{V}^{(2)}} V_i^{(2)})) / \mathbb{F}(G, \mathbf{q}, n - 2) \\ &\leq 2\epsilon_1 \|\mathbf{q}\|_1 / \mathbb{F}(G, \mathbf{q}, n - 2) = 2(\frac{\epsilon q_{\min}}{3n})^2. \end{aligned} \quad (14)$$

Recall that A_V is the corresponding classical event of V defined in the proof of Lemma 4.1, and $\mathbb{F}(G, \mathbf{p}, n - 2)$ monotonically decreases as \mathbf{p} increases.

Now, we are going to construct an exclusive subspace set $\mathcal{V}^{(3)} \sim G_B$ with $R(\mathcal{V}^{(3)}) \geq (1 - \epsilon)\mathbf{q}$ from $\mathcal{V}^{(2)}$, which concludes the proof coupled with Lemma A.4. For simplicity, we use \mathcal{H}_i and V_i to represent $\mathcal{H}_i^{(2)}$ and $V_i^{(2)}$ respectively.

For any i, j where $i \in \Gamma_j$, according to the structure lemma, $\mathcal{H}_{\mathcal{N}(i) \cap \mathcal{N}(j)}$ can be decomposed to some orthogonal subspaces $\mathcal{H}_{\mathcal{N}(i) \cap \mathcal{N}(j)} = \bigoplus_k W_k = \bigoplus_k W_{k1} \otimes W_{k2}$ s.t.

1. $V_i^{loc} = \bigoplus_k V_i|_{W_{k1}} \otimes W_{k2}$, where $V_i|_{W_{k1}} \subseteq \mathcal{H}_{\mathcal{N}(i) \setminus \mathcal{N}(j)} \otimes W_{k1}$.
2. $V_j^{loc} = \bigoplus_k V_j|_{W_{k2}} \otimes W_{k1}$, where $V_j|_{W_{k2}} \subseteq \mathcal{H}_{\mathcal{N}(j) \setminus \mathcal{N}(i)} \otimes W_{k2}$.

For simplicity, let $X_{ijk} = V_i|_{W_{k1}} \otimes W_{k2} \otimes \mathcal{H}_{[n] \setminus \mathcal{N}(i)}$, $X_{jik} = V_j|_{W_{k2}} \otimes W_{k1} \otimes \mathcal{H}_{[n] \setminus \mathcal{N}(j)}$, and $Y_{ijk} = W_k \otimes \mathcal{H}_{[n] \setminus (\mathcal{N}(i) \cap \mathcal{N}(j))}$. Thus $\bigoplus_k X_{ijk} = V_i$, $\bigoplus_k X_{jik} = V_j$ and $\bigoplus_k Y_{ijk} = \mathcal{H}$. For any $i \in [m]$, we define $V_i^{(3)}$ as the orthogonal complement of $\sum_{j: j \in \Gamma_i, k: R(X_{ijk}) \leq R(X_{jik})} X_{ijk}$ in V_i . Since X_{ijk} only depends on $\mathcal{H}_{\mathcal{N}(i)}$, so does $V_i^{(3)}$. Therefore, $\mathcal{V}^{(3)} \sim G_B$.

$\mathcal{V}^{(3)}$ is commuting and exclusive: for any given i, j where $i \in \Gamma_j$, it is not hard to see $V_i^{(3)}$ is a subspace of $\bigoplus_{k: R(X_{ijk}) > R(X_{jik})} Y_{ijk}$ and $V_j^{(3)}$ is a subspace of $\bigoplus_{k: R(X_{jik}) > R(X_{ijk})} Y_{jik}$, thus $V_i^{(3)}$ and $V_j^{(3)}$ are orthogonal, therefore commuting and exclusive.

In the following, we will finish the proof by showing $R(\mathcal{V}^{(3)}) \geq (1 - \epsilon)\mathbf{q}$. Since V_{ijk} and V_{jik} are R -independent in Y_{ijk} , we have

$$R(X_{ijk} \cap X_{jik}) = R(X_{ijk} \cap X_{jik} | Y_{ijk}) \cdot R(Y_{ijk}) = R(X_{ijk} | Y_{ijk}) R(X_{jik} | Y_{ijk}) R(Y_{ijk}). \quad (15)$$

Thus,

$$\begin{aligned} R(V_i \cap V_j) &= R(\bigoplus_k X_{ijk} \cap X_{jik}) = \sum_k R(X_{ijk} \cap X_{jik}) \\ &= \sum_k R(X_{ijk} | Y_{ijk}) R(X_{jik} | Y_{ijk}) R(Y_{ijk}) \\ &\geq \sum_k (\min\{R(X_{ijk} | Y_{ijk}), R(X_{jik} | Y_{ijk})\})^2 R(Y_{ijk}) \\ &= \mathbb{E}_k[(\min\{R(X_{ijk} | Y_{ijk}), R(X_{jik} | Y_{ijk})\})^2]. \end{aligned} \quad (16)$$

Meanwhile, we also have

$$\begin{aligned} \sum_k \min\{R(X_{ijk}), R(X_{jik})\} &= \sum_k \min\{R(X_{ijk} | Y_{ijk}), R(X_{jik} | Y_{ijk})\} R(Y_{ijk}) \\ &= \mathbb{E}_k(\min\{R(X_{ijk} | Y_{ijk}), R(X_{jik} | Y_{ijk})\}). \end{aligned}$$

By Jensen's inequality, we have

$$\begin{aligned} R(V_i \cap V_j) &\geq \mathbb{E}_k[(\min\{R(X_{ijk} | Y_{ijk}), R(X_{jik} | Y_{ijk})\})^2] \\ &\geq [\mathbb{E}_k(\min\{R(X_{ijk} | Y_{ijk}), R(X_{jik} | Y_{ijk})\})]^2 \\ &= (\sum_k \min\{R(X_{ijk}), R(X_{jik})\})^2. \end{aligned} \quad (17)$$

Thus, for any i ,

$$\begin{aligned} &\mathbb{R}(\sum_{j: j \in \Gamma_i} \bigoplus_{k: R(X_{ijk}) \leq R(X_{jik})} X_{ijk}) \\ &\leq \sum_{j: j \in \Gamma_i} \sum_{k: R(X_{ijk}) \leq R(X_{jik})} R(X_{ijk}) \\ &\leq \sum_{j: j \in \Gamma_i} \sum_k \min\{R(X_{ijk}), R(X_{jik})\} \\ &\leq \sum_{j: j \in \Gamma_i} (R(V_i \cap V_j))^{1/2} \quad (\text{by (17)}) \\ &\leq \sum_{j: j \in \Gamma_i} (2(\frac{\epsilon q_{min}}{3n})^2)^{1/2} \quad (\text{by (14)}) \\ &\leq n(2(\frac{\epsilon q_{min}}{3n})^2)^{1/2} < 2\epsilon q_{min}/3. \end{aligned}$$

Recall $V_i^{(3)}$ is the orthogonal complement of $\sum_{j:j \in \Gamma_i} \bigoplus_{k:R(X_{ijk}) \leq R(X_{jik})} X_{ijk}$ in space V_i , we have

$$\begin{aligned} R(V_i^{(3)}) &= R(V_i) - R(\sum_{j:j \in \Gamma_i} \bigoplus_{k:R(X_{ijk}) \leq R(X_{jik})} X_{ijk}) \\ &\geq (1 - \epsilon_1)q_i - 2\epsilon q_{min}/3 \geq (1 - \epsilon/3)q_i - 2\epsilon q_{min}/3 \geq (1 - \epsilon)q_i. \end{aligned}$$

Therefore, we have $R(\mathcal{V}^{(3)}) \geq (1 - \epsilon)\mathbf{q}$. \square

Now we are ready to give the proof of our main theorem of this appendix.

Theorem 4.9. *Given an interaction bipartite graph G_B and a vector \mathbf{r} of positive reals, the following two conditions are equivalent:*

1. *For any rational $\lambda \mathbf{r} \in \mathcal{CI}(G_B) \setminus \mathcal{C}\partial(G_B)$, there is an exclusive subspace set with interaction bipartite graph G_B and relative dimension vector $\lambda \mathbf{r}$.*
2. *G_B is gapless for CLLL in the direction of \mathbf{r} .*

Proof. (1 \Rightarrow 2): Arbitrarily fix $\lambda > 0$ such that $\mathbf{q} \triangleq \lambda \mathbf{r} \in \mathbb{I}(G_B)$ and \mathbf{q} is rational. Let $\mathcal{V} \sim G_B$ be an exclusive subspace set such that $R(\mathcal{V}) = \mathbf{q}$ and $R(\sum_{V \in \mathcal{V}} V) < 1$. Recall in the proof of Lemma 4.1, A_V is the classical event corresponding to V , and $A_V \sim G_D(G_B)$, $\mathbb{P}(A_{V_i}) = q_i$, $\mathbb{P}(\cup_i A_{V_i}) = R(\sum_i V_i) < 1$. In addition, A_{V_i} 's are exclusive, thus according to Lemma A.2, $\mathbf{q} \in \mathcal{I}(G_B)$, which means H is gapless in the direction of \mathbf{r} .

(2 \Rightarrow 1): It is immediate by Lemma A.5. \square

The following corollary is immediate by Theorem 4.9 and Lemma A.4. By this corollary, one can prove gaplessness just by constructing a commuting subspace set, without computing the critical threshold of CLLL or Shearer's bound.

Corollary 4.10. *Given an interaction bipartite graph G_B and a rational vector $\mathbf{r} \in \mathcal{C}\partial(G_B)$, if there exists an exclusive subspace set with interaction bipartite graph G_B and relative dimension vector \mathbf{r} , then G_B is gapless in the direction of \mathbf{r} .*

B Proof of Theorem 4.13

Proof. (Duplicate- L -Vertex, Duplicate- R -Vertex): The proofs of Duplicate- L -Vertex and Duplicate- R -Vertex are similar to that in [24]. Moreover, let G'_B be the resulting graph by applying Duplicate- R -Vertex to G_B , we have $\mathcal{C}\partial(G'_B) = \mathcal{C}\partial(G_B)$.

(Delete- R -Leaf): Suppose vertex $n + 1$ is added to the right side, if $\mathcal{N}(n + 1)$ is empty, it's trivial, otherwise assume $\mathcal{N}(n + 1) = \{m\}$ and $G'_B = ([m], [n + 1], E'_B)$ is the resulted bipartite graph. Note that the base graph G_D remains unchanged, it suffices to prove $\mathcal{C}\partial(G_B) = \mathcal{C}\partial(G'_B)$. It is easy to see $\mathcal{C}\partial(G'_B) \subseteq \mathcal{CI}(G_B) \cup \mathcal{C}\partial(G_B)$, so it remains to show $\mathcal{C}\partial(G_B) \subseteq \mathcal{CI}(G'_B) \cup \mathcal{C}\partial(G'_B)$.

Consider another interaction bipartite graph $G''_B = ([m], [n + 1], E''_B)$ obtained by applying the inverse operation of Delete-Edge to G'_B : $\forall i \in \mathcal{N}(n)$, we add the edge $(i, n + 1)$. On one hand, it is easy to see $\mathcal{C}\partial(G''_B) \subseteq \mathcal{CI}(G'_B) \cup \mathcal{C}\partial(G'_B)$. On the other hand, note that $\mathcal{N}(n) = \mathcal{N}(n + 1)$, thus G''_B can be viewed as the resulting bipartite graph by applying Duplicate- R -Vertex to G_B , we have $\mathcal{C}\partial(G''_B) = \mathcal{C}\partial(G_B)$.

(Delete- L -Leaf): Let $G_B = ([m], [n], E)$. Suppose vertex $m+1$ is added to the left side, if $\mathcal{N}(m+1)$ is empty, it's trivial. Otherwise, w.l.o.g., assume $\mathcal{N}(m+1) = \{n\}$ and let $G'_B = ([m+1], [n], E_B \cup \{(m+1, n)\})$ be the resulted bipartite graph. Besides, assume $\mathcal{N}(n) = \{1, 2, \dots, k, m+1\}$.

G_B is gapless $\Rightarrow G'_B$ is gapless: By Theorem 4.9, it suffices to show for any rational $\mathbf{r}' \triangleq (r'_1, \dots, r'_{m+1}) \in \mathcal{CI}(G'_B) \setminus \mathcal{CD}(G'_B)$, there is such an exclusive subspace set \mathcal{V}' . Let $\mathbf{r} = (\frac{r'_1}{1-r'_{m+1}}, \frac{r'_2}{1-r'_{m+1}}, \dots, \frac{r'_k}{1-r'_{m+1}}, r'_{k+1}, \dots, r'_m)$. First, we claim that $\mathbf{r} \in \mathcal{CI}(G_B) \setminus \mathcal{CD}(G_B)$.

This is because otherwise, by the definition of $\mathcal{CD}(G_B)$, there is a commuting subspace set $\mathcal{V} \sim G_B$ with $R(\mathcal{V}) = (1 + \epsilon)\mathbf{r}$ satisfying $R(\sum_{V \in \mathcal{V}} V) = 1$ for any rational $\epsilon > 0$. We construct a commuting subspace set $\mathcal{V}' \sim G'_B$ as follows: suppose $1 - r'_{n+1} = a/b$ where a and b are integers. Let $\mathcal{H}'_n = \mathcal{H}_n \otimes \mathcal{H}_n^c$ where $\dim(\mathcal{H}_n^c) = b$, and $\mathcal{H}'_i = \mathcal{H}_i$ for any $i < n$. Then let

- If $i \in [k]$, $V'_i = V_i \otimes Y$, where Y can be any subspace of \mathcal{H}_n^c with dimension a .
- If $k < i \leq n$, $V'_i = V_i \otimes \mathcal{H}_n^c$.
- If $i = n + 1$, $V'_i = Y^\perp \otimes \bigotimes_{i=1}^n \mathcal{H}_i$

It is easy to verify that \mathcal{V}' is commuting, $\mathcal{V}' \sim G'_B$, $R(\sum_{V' \in \mathcal{V}'} V') = 1$ and $R(\mathcal{V}') = ((1+\epsilon)r'_1, \dots, (1+\epsilon)r'_n, r'_{n+1}) \leq (1 + \epsilon)\mathbf{r}'$. Thus, by Lemma 4.2 there is also a commuting subspace set $\mathcal{V}'' \sim G'_B$ with relative dimensions $(1 + \epsilon)\mathbf{r}'$ such that $R(\sum_{V'' \in \mathcal{V}''} V'') = 1$, a contradiction.

since $\mathbf{r} \in \mathcal{CI}(G_B) \setminus \mathcal{CD}(G_B)$ and G_B is gapless, by Theorem 4.9 we have there is an exclusive subspace set \mathcal{V} with interaction bipartite graph G_B and relative dimensions \mathbf{r} . Then it is easy to verify that the commuting subspace set \mathcal{V}' constructed above is also exclusive, $\mathcal{V}' \sim G'_B$, and $R(\mathcal{V}') = \mathbf{r}'$.

G'_B is gapless $\Rightarrow G_B$ is gapless: By Theorem 4.9, it suffices to show there is an exclusive $\mathcal{V} \sim G_B$ for any $\mathbf{r} \in \mathcal{CI}(G_B) \setminus \mathcal{CD}(G_B)$. Let $(1 + \lambda)\mathbf{r} \in \mathcal{CD}(G_B)$ be the vector on the boundary, and $\mathbf{r}' = (\mathbf{r}, \epsilon)$ where $\epsilon < \lambda r_1$. It is not hard to see that $\mathbf{r}' \in \mathcal{CI}(G'_B) \setminus \mathcal{CD}(G'_B)$. And then, by Theorem 4.9, there is an exclusive subspace set $\mathcal{V}' = \{V'_1, \dots, V'_{m+1}\}$ with interaction bipartite graph G'_B and relative dimensions \mathbf{r}' . Let $\mathcal{V} = \{V'_1, V'_2, \dots, V'_m\}$, it is easy to see that \mathcal{V} is exclusive, $\mathcal{V} \sim G_B$ and $R(\mathcal{V}) = \mathbf{r}$. \square